EECS 216
Signals and Systems

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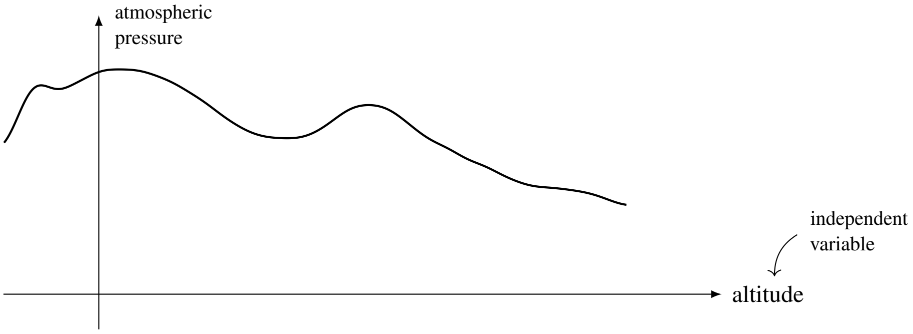
Contents

1. Signals

In this course a signal will be a real or complex function of one independent variable, e.g.

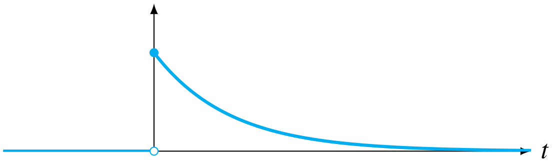


Usually this **independent variable** will be taken to be **time**.

1.1 Continuous-time vs. Discrete-time Signals

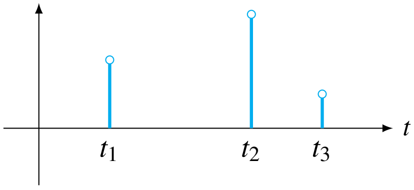
- A signal defined at all instants in time is known as a ***continuous-time signal***

  e.g. $x(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$



- A ***discrete-time signal*** is defined only at a finite (or countable) number of time instants

  e.g.



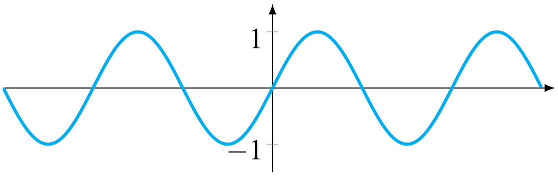
  e.g. $x(t) = \sin(t)$ for $t = n\Delta,\ n \in \{0, \pm 1, \pm 2, \dots\}$, for other values of t, $x(t)$ is undefined

We will mainly deal with continuous-time signals in this course, but we will also see discrete-time signals (*sampling*).

1.2 Analog vs. Digital Signals

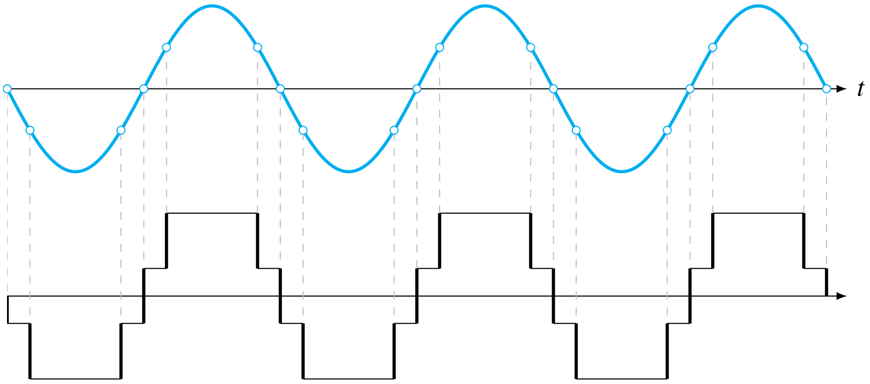
- If $x(t)$ takes (or can take) a continuum of values, then it is said to be ***analog***
- If $x(t)$ takes (or can take) only a finite set of values, then it is said to be ***digital***

  e.g. $x(t) = \sin(t)$



takes all values in $[-1,1]$, so it is analog.

e.g. $x(t)=\begin{cases}-3 & \text{if } -1\le \sin t\le -\frac{1}{2}\\ -1 & \text{if } -\frac{1}{2}<\sin t\le 0\\ 1 & \text{if } 0<\sin t\le \frac{1}{2}\\ 3 & \text{if } \frac{1}{2}<\sin t\le 1\end{cases}$ can only take values in $\{-3,-1,1,3\}$ so it is digital.



1.3 Periodic vs. Non-periodic Signals

- $x(t)$ is said to be periodic if there exists a number $T>0$ such that

$$x(t+nT)=x(t) \tag{1.1}$$

for all $n\in\{0,\pm1,\pm2,\dots\}$.

- The smallest positive value of T for which this is true is called the ***period***.
  e.g. $x(t)=3\cos\left(2t+\frac{\pi}{3}\right)$ for all t is periodic with period $T=\frac{2\pi}{2}=\pi$.
- The frequency of a periodic signal is defined as $f=\frac{1}{T}$ (in Hz).
  $\left(\text{angular frequency } \omega=2\pi f=\frac{2\pi}{T} \text{ in radians/sec}\right)$
- All signals that are not periodic are called non-periodic.

1.4 Even and Odd signals

- $x(t)$ is even if and only if $x(-t)=x(t)$.
- $x(t)$ is odd if and only if $x(-t)=-x(t)$.

1.4.1 Decomposition

Proposition 1.1 Every function $x(t)$ can be written as a sum of an even function $x_e(t)$ and an odd function $x_o(t)$.

Proof. Define:

$$x_e(t)=\frac{x(t)+x(-t)}{2} \tag{1.2}$$

$$x_o(t)=\frac{x(t)-x(-t)}{2} \tag{1.3}$$

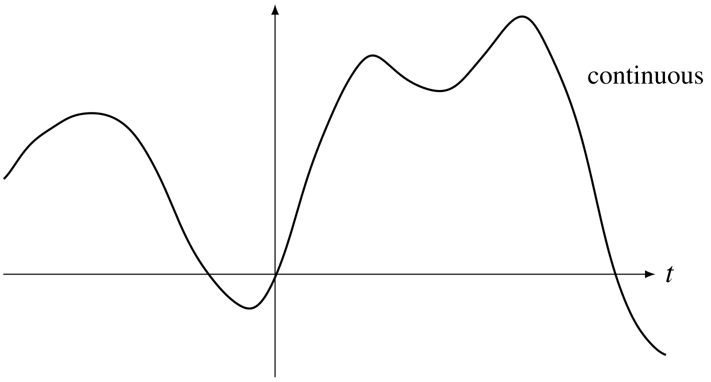
$x_e(t)$ is even: $x_e(-t) = \frac{x(-t)+x(t)}{2} = \frac{x(t)+x(-t)}{2} = x_e(t)$
$x_o(t)$ is odd: $x_o(-t) = \frac{x(-t)-x(t)}{2} = -\frac{x(t)-x(-t)}{2} = -x_o(t)$

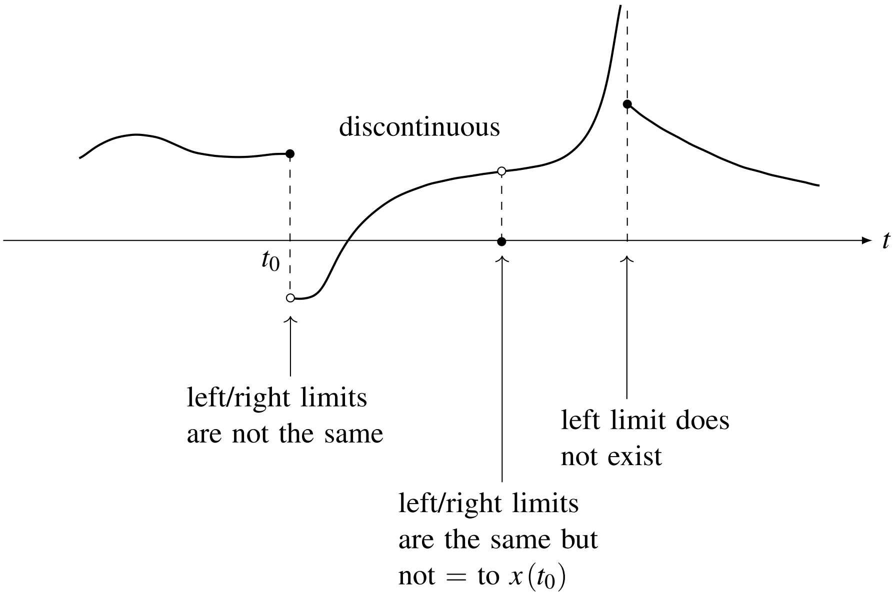
$$x_e(t)+x_o(t) = \frac{x(t)+x(-t)}{2} + \frac{x(t)-x(-t)}{2} = \frac{2x(t)}{2} = x(t)$$

$\square$

1.5 Continuous and Piecewise Continuous Signals

Idea of a continuous function:
you can draw its graph without lifting your pencil from the page.





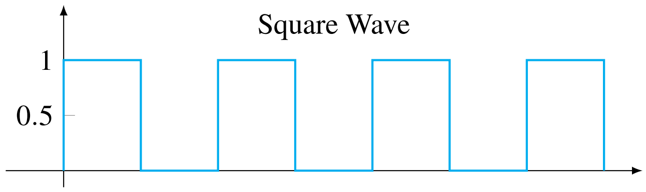
Notation:

$$x(t_0^-) = \lim_{t \uparrow t_0} x(t) \quad \text{from below (left)} \tag{1.4}$$

$$x(t_0^+) = \lim_{t \downarrow t_0} x(t) \quad \text{from above (right)} \tag{1.5}$$

Definition 1.1 $x(t)$ is piecewise continuous if for any bounded interval $[t_1, t_2]$ there are at most a finite number of discontinuities **AND** at these points the left and right limits exist.

e.g.



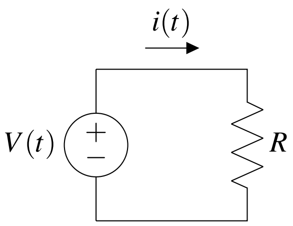
1.6 Energy and Power Signals

Define the following quantities:

(a) Instantaneous power

$$P_{inst} = |x(t)|^2 \quad \text{(magnitude to deal with complex signals)} \tag{1.6}$$

Intuition from physics:



Inst. power dissipated $= Ri^2(t) = \frac{V^2(t)}{R}$

(b) Energy over an interval $[t_1, t_2]$:

$$E_{t_1,t_2} = \int_{t_1}^{t_2} P_{inst}(t)dt = \int_{t_1}^{t_2} |x(t)|^2 dt \tag{1.7}$$

(c) Total energy:

$$E_{tot} = \int_{-\infty}^{\infty} P_{inst}(t)dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{(be careful it may be } \infty) \tag{1.8}$$

(d) Average power over an interval $[t_1, t_2]$:

$$P_{avg,t_1,t_2} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P_{inst}(t)dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \tag{1.9}$$

(e) Average power:

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} P_{inst}(t)dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{1.10}$$

Note: P_{avg} is easy to evaluate for periodic signals with period T_0:

$$P_{avg} = \frac{1}{T_0} \int_{<T_0>} P_{inst}(t)dt = \frac{1}{T_0} \int_{<T_0>} |x(t)|^2 dt \tag{1.11}$$

where $< T_0 >$ means over any interval of width T_0, i.e. $\int_{\alpha}^{\alpha + T_0}$, $\forall \alpha$ (for any α).

Definition 1.2 An ***energy signal*** is a signal with *finite* total energy, i.e.

$$E_{tot} < \infty$$

Definition 1.3 A ***power signal*** is a signal with *finite*, *non-zero* average power, i.e.

$$0 < P_{avg} < \infty$$

R Textbook uses slightly different definition: power signal $\Leftrightarrow P_{avg} < \infty$ and $E_{tot} = \infty$. These definitions **are NOT** equivalent.

■ **Example 1.1** Consider the signal $x(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$. Is it an energy signal or a power signal?

$$E_{tot} = \int_{-\infty}^{\infty} x^2(t)dt = \int_0^{\infty} e^{-2t}dt = \left.\frac{e^{-2t}}{-2}\right|_0^{\infty} = 0 - \frac{1}{-2} = \frac{1}{2} < \infty \implies \text{this is an energy signal.}$$

$P_{avg} = ---?$ (not in class)

$$\int_{-T}^{T} x^2(t)dt = \int_0^{T} e^{-2t}dt = \left.\frac{e^{-2t}}{-2}\right|_0^{T} = \frac{e^{-2T}-1}{-2} = \frac{1-e^{-2T}}{2}$$

$$\therefore\ P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\frac{1-e^{-2T}}{2} = \lim_{T\to\infty} \frac{1-e^{-2T}}{4T} = 0$$

■

■ **Example 1.2** Consider the sinusoidal signal $x(t) = \sin(\omega_0 t)$. Is it an energy signal or a power signal?

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin^2(\omega_0 t)dt$$

Using the identity $\cos 2\alpha = 1 - 2\sin^2\alpha$, $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$, so

$$\begin{aligned}\int_{-T}^{T} \sin^2(\omega_0 t)dt &= \int_{-T}^{T}\frac{1}{2}dt - \frac{1}{2}\int_{-T}^{T}\cos(2\omega_0 t)dt = T - \left.\frac{1}{2}\frac{\sin(2\omega_0 t)}{2\omega_0}\right|_{-T}^{T} \\ &= T - \frac{\sin(2\omega_0 T) - \sin(-2\omega_0 T)}{4\omega_0} = T - \frac{\cancel{2}\sin(2\omega_0 T)}{{}_2\cancel{4}\omega_0} = T - \frac{\sin(2\omega_0 T)}{2\omega_0}\end{aligned}$$

Therefore

$$P_{avg} = \lim_{T\to\infty}\frac{1}{2T}\left[T - \frac{\sin(2\omega_0 T)}{2\omega_0}\right] = \lim_{T\to\infty}\left[\frac{1}{2} - \frac{\sin(2\omega_0 T)}{2\omega_0 T}\right] = \frac{1}{2}$$

(not in class)
We could have also found P_{avg} as:

$$P_{avg} = \frac{1}{T_0}\int_0^{T_0}\sin^2(\omega_0 t)dt$$

where $\omega_0 = \frac{2\pi}{T_0} \Leftrightarrow T_0 = \frac{2\pi}{\omega_0}$ since $x(t)$ is periodic.

$$\begin{aligned} P_{avg} &= \frac{1}{T_0}\int_0^{T_0} \frac{1-\cos(2\omega_0 t)}{2} dt = \frac{1}{T_0}\left[\frac{T_0}{2} - \frac{\sin(2\omega_0 t)}{4\omega_0}\Big|_0^{T_0}\right] = \frac{1}{T_0}\left[\frac{T_0}{2} - \frac{\sin(2\omega_0 T_0)}{4\omega_0}\right] \\ &= \frac{1}{2} - \frac{\sin(2\omega_0 T_0)}{4\omega_0 T_0} \end{aligned}$$

Since $\omega_0 = \frac{2\pi}{T_0}$, $\omega_0 T_0 = 2\pi$, so

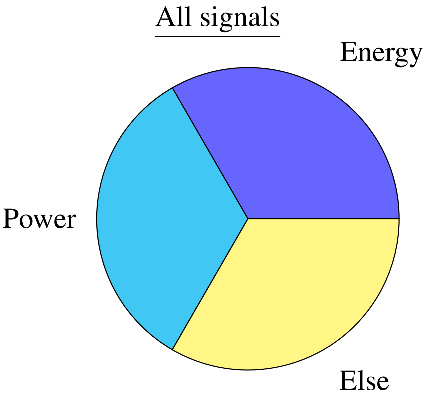
$$P_{avg} = \frac{1}{2} - \frac{\sin(4\pi)}{8\pi} = \frac{1}{2} - 0 = \frac{1}{2}$$

$0 < P_{avg} = \frac{1}{2} < \infty$, so $x(t)$ is a power signal.

$$\begin{aligned} E_{tot} &= \cdots \text{ (not in class)} \\ &\vdots \\ &\text{(show it)} \\ &\vdots \\ &= \infty \end{aligned}$$

■

One can show that:



In other words, a signal $x(t)$ is either energy or power or none of the above, but it *cannot* be energy and power at the same time.

Proof. (not in class)
We will show that if $x(t)$ is an energy signal ($E_{tot} < \infty$) then its average power $P_{avg} = 0$ so it cannot be a power signal.

Indeed $E_{tot} = M < \infty$, where M is some finite number.

$$
\begin{gathered}
E_{tot} = \int_{-\infty}^{\infty} |x(t)|^2 dt = M < \infty \\
\Downarrow \\
\int_{-T}^{T} |x(t)|^2 dt \leq \int_{-\infty}^{\infty} |x(t)|^2 dt = M \\
\Downarrow \\
\frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \leq \frac{M}{2T} \\
\Downarrow \text{(take limit as } T \to \infty) \\
P_{avg} \leq \lim_{T\to\infty} \frac{M}{2T} = 0 \\
\Downarrow \\
P_{avg} = 0
\end{gathered}
$$

$\square$

Q: Can you think of a signal which is neither power nor energy?
Hint: It has to be big!

A: $x(t) = t$

$$
\begin{aligned}
\int_{-T}^{T} x^2(t)dt = \int_{-T}^{T} t^2 dt = \frac{t^3}{3}\bigg|_{-T}^{T} &= \frac{2T^3}{3} \\
E_{tot} = \lim_{T\to\infty} \frac{2T^3}{3} &= \infty \\
P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\frac{2T^3}{3} = \lim_{T\to\infty} \frac{T^2}{3} &= \infty
\end{aligned}
$$

In fact $x(t) = t^{\alpha}$, $\forall \alpha > 0$ (for any $\alpha > 0$) has $E_{tot} = P_{avg} = \infty$.

R Textbook definition and the one presented here are not equivalent, i.e.

$$0 < P_{avg} < \infty \not\Longleftrightarrow P_{avg} < \infty \;\&\; E_{tot} = \infty$$

i.e. if $P_{avg} < \infty$ then $P_{avg} > 0 \not\Longleftrightarrow E_{tot} = \infty$ ($\Longrightarrow$ is true, same proof as before). The problem lies in that if $P_{avg} < \infty$, $E_{tot} = \infty \not\Longrightarrow P_{avg} > 0 \Longleftrightarrow P_{avg} = 0 \not\Longrightarrow E_{tot} < \infty$. Indeed, we can find signals with $P_{avg} = 0$ and $E_{tot} = \infty$. e.g. $x(t) = \begin{cases} \frac{1}{t^{\alpha}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$ for $0 < \alpha < \frac{1}{2}$.

Note:

- A signal is "bounded" if there exists a finite constant M such that $|x(t)| < M$, $\forall t$ (for all t).
- A signal is of "finite duration" if there exist finite constants t_1, t_2 such that $x(t) = 0$ if $t \notin [t_1, t_2]$.

One can show that

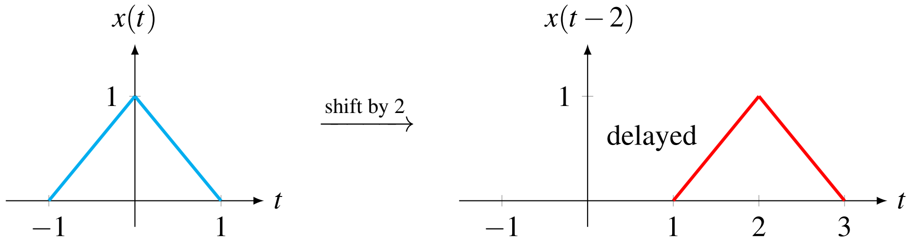
- Every **bounded**, **finite duration** signal is an energy signal
- Every **bounded**, **periodic** signal is a power signal

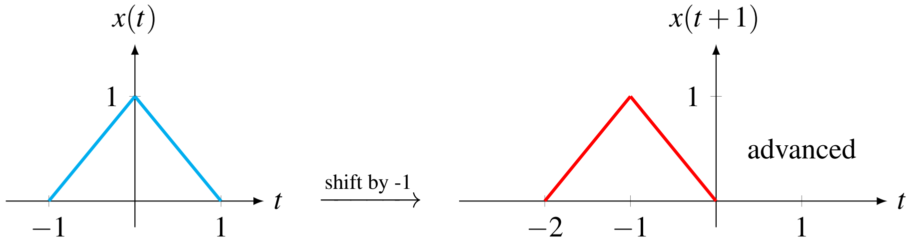
1.7 Transformation of the Independent Variable

Shifting, Reflecting, and Time Scaling.

1.7.1 Shifting by t_0

$$x(t) \xrightarrow{\text{shift by } t_0} y(t) = x(t-t_0)$$

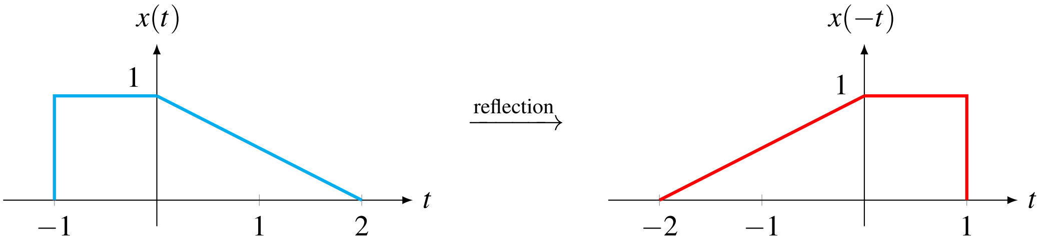




- if $t_0 > 0$, delay
- if $t_0 < 0$, advance (not physically realizable)

1.7.2 Reflecting

$$x(t) \xrightarrow{\text{reflection}} y(t) = x(-t)$$



Shifting and reflecting do **NOT** commute:

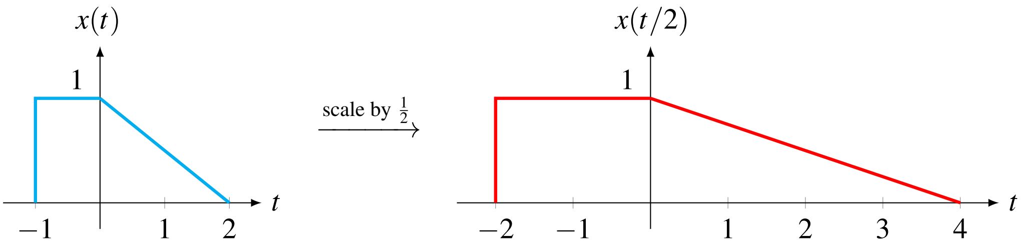
$$x(t) \xrightarrow{\text{shift by } t_0} y(t) = x(t-t_0) \xrightarrow{\text{reflection}} y(-t) = x(-t-t_0)$$
$$x(t) \xrightarrow{\text{reflection}} z(t) = x(-t) \xrightarrow{\text{shift by } t_0} z(t-t_0) = x(-(t-t_0)) = x(-t+t_0)$$

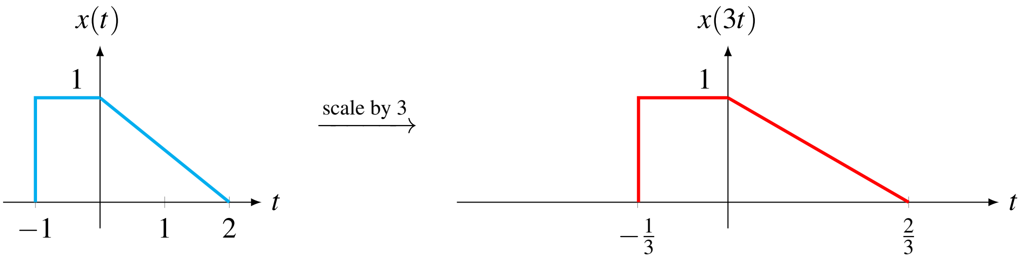
You have to shift by $-t_0$!

1.7.3 Time Scaling

$$x(t) \xrightarrow{\text{scale by } n} y(t) = x(nt)$$

scaling by $n \iff$ scaling by $|n|$ and possible reflection if $n < 0$.





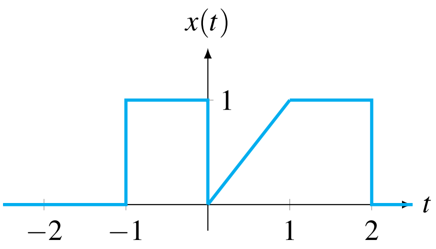
$|n| < 1 \implies$ expansion (slow motion)
$|n| > 1 \implies$ compression (fast forward)

R Scaling and shifting do **NOT** commute:

$$x(t) \xrightarrow{\text{shift by } t_0} y(t) = x(t - t_0) \xrightarrow{\text{scale by } n} y(nt) = x(nt - t_0)$$
$$x(t) \xrightarrow{\text{scale by } n} z(t) = x(nt) \xrightarrow{\text{shift by } t_0} z(t - t_0) = x(n(t - t_0)) = x(nt - nt_0)$$

Note: For $n < 0$, $n = -|n|$, so $x(nt) = x(-|n|t)$. The $-$ is responsible for reflection, and the $|n|$ is responsible for scaling.

■ **Example 1.3** Suppose $x(t)$ is given by the plot below. Define $y(t) = x\left(1 - \frac{t}{2}\right)$. Plot $y(t)$.

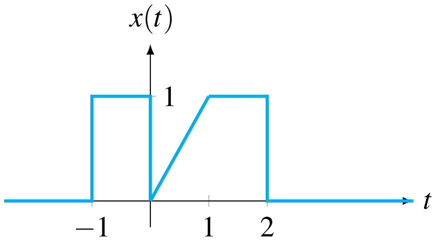


Two methods to solve this problem:

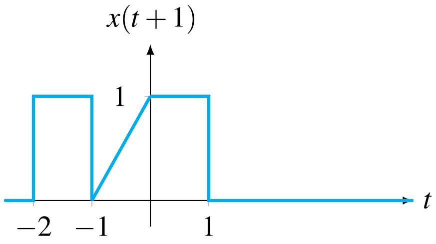
- **Method 1:** View the signal $y(t)$ as $y(t) = x(at - b)$. In our example, $a = -\frac{1}{2}$ and $b = -1$. This suggests shifting and then scaling:

$$x(t) \xrightarrow{\text{shift by } b} x(t - b) \xrightarrow{\text{scale by a}} x(at - b)$$

Original:

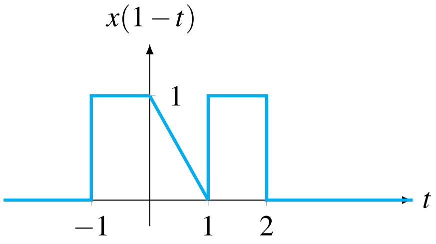


Shift by $b = -1$ (advance)

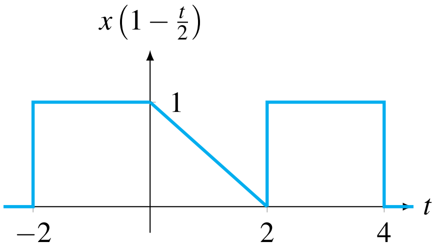


Scale by $a = -\frac{1}{2}$:

Reflect



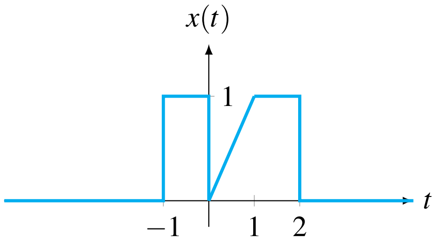
Scale by $|a| = \frac{1}{2}$:



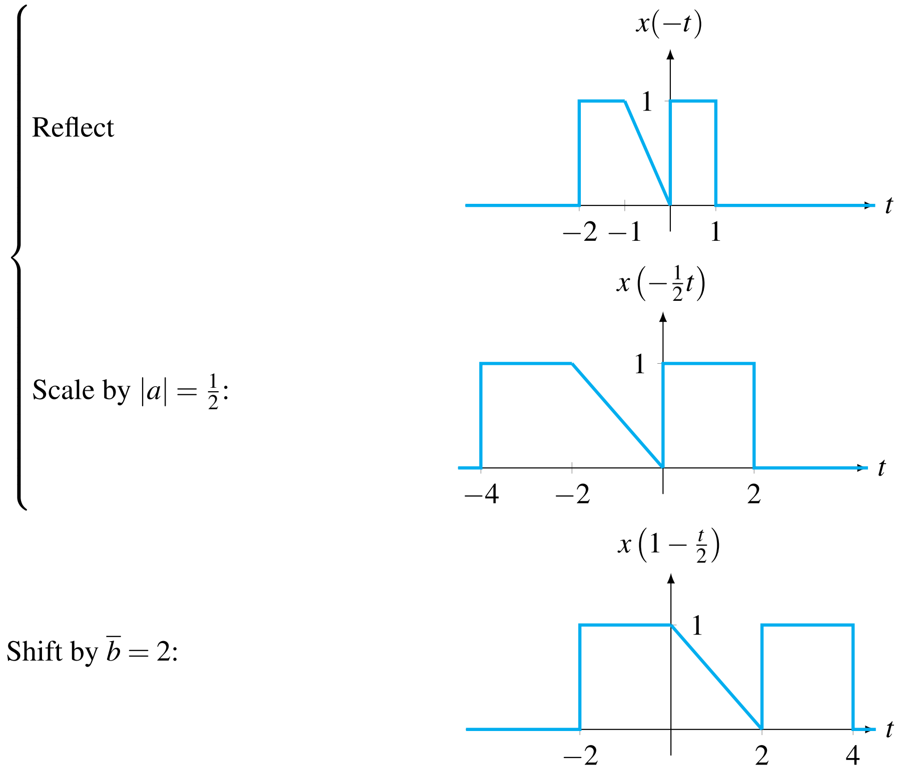
- **Method 2:** View the signal $y(t)$ as $y(t) = x\left(\bar{a}(t-\bar{b})\right)$. In our example, $x\left(1-\frac{t}{2}\right) = x\left(\left(-\frac{1}{2}\right)(-2+t)\right) = x\left(\left(-\frac{1}{2}\right)(t-2)\right)$ so $\bar{a} = -\frac{1}{2}$, $\bar{b} = 2$. This suggests scaling and then shifting.

$$x(t) \xrightarrow{\text{scale by } \bar{a}} x(\bar{a}t) \xrightarrow{\text{shift by } \bar{b}} x(\bar{a}(t-\bar{b}))$$

Original:



Scale by $a = -\frac{1}{2}$:



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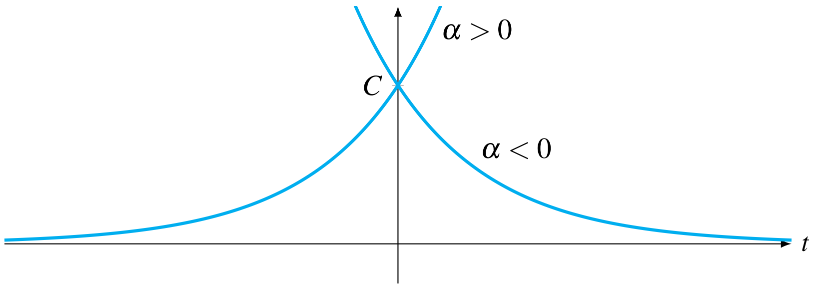
1.8 Important Engineering Signals

1. **Real Exponential:**

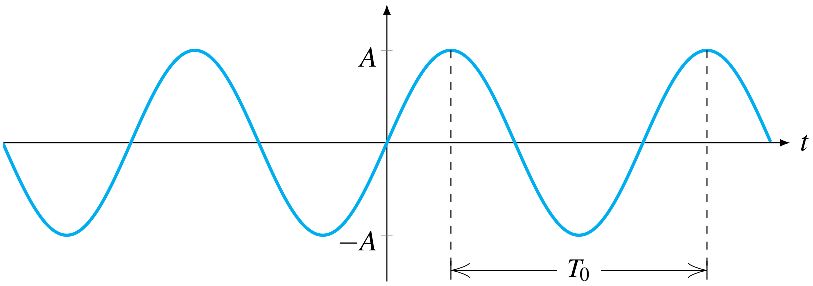
$$x(t) = Ce^{\alpha t} \tag{1.12}$$



2. **Sinusoid:**

$$x(t) = A\sin(\omega_0 t + \phi) \tag{1.13}$$

, where ω_0 is the angular frequency, $\omega_0 = 2\pi f_0$, f_0 is the frequency, and $f_0 = \frac{1}{T_0}$, T_0 is the period, and ϕ is the phase shift.

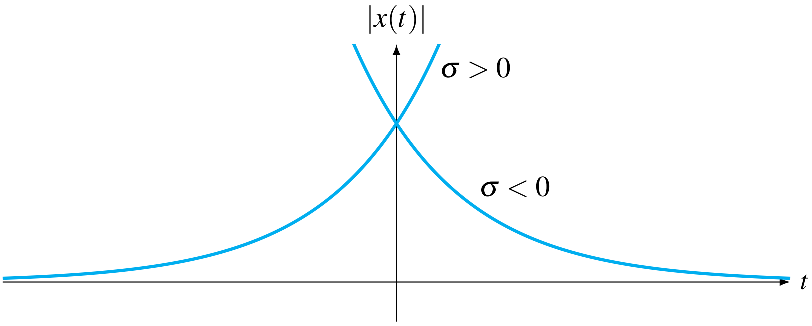


3. **Complex Exponential:**

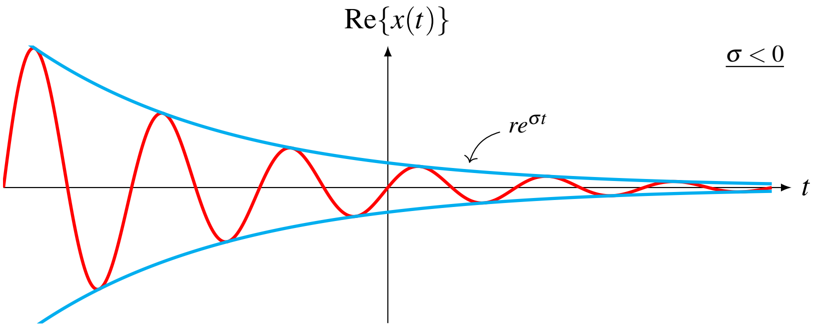
$$x(t) = Ce^{st} \tag{1.14}$$

, where C and s are complex numbers.

$$\begin{aligned} C &= re^{j\theta} \\ s &= \sigma + j\omega \\ x(t) &= re^{j\theta}e^{(\sigma+j\omega)t} = re^{j\theta}e^{\sigma t}e^{j\omega t} = (re^{\sigma t})e^{j(\omega t+\theta)} \\ |x(t)| &= re^{\sigma t} \end{aligned}$$



$$\begin{aligned} \mathrm{Re}\{x(t)\} &= re^{\sigma t}\cos(\omega t + \theta) \\ \mathrm{Im}\{x(t)\} &= re^{\sigma t}\sin(\omega t + \theta) \end{aligned}$$



4. **Unit Step:**

$$u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{1.15}$$

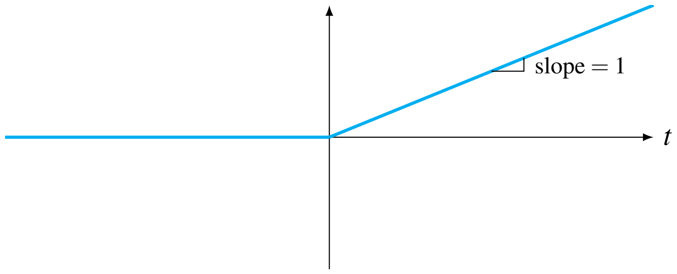
We will **NOT** define $u(t)$ at $t = 0$. Sometimes it is defined as $u(0) = \frac{1}{2}$, but it will not make a difference in our results.

5. **Signum:**

$$sgn(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases} = 2u(t) - 1 \tag{1.16}$$

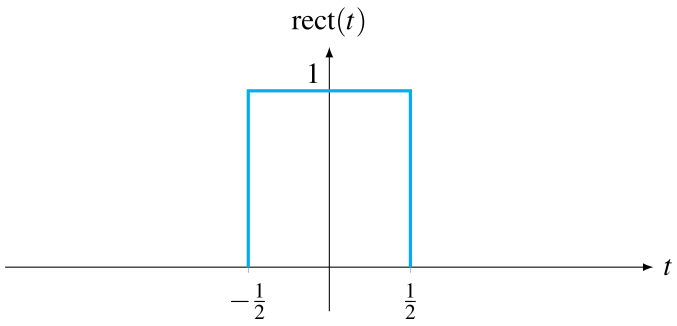
6. **Unit Ramp:**

$$r(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} = tu(t) \tag{1.17}$$

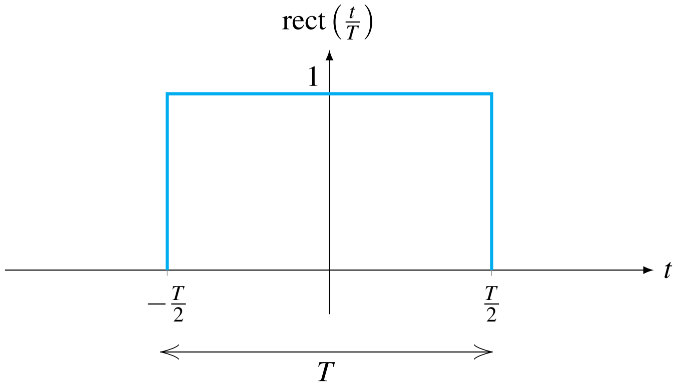


7. **Rectangular Pulse:**

$$\text{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.18}$$



$\text{rect}\left(\frac{t}{T}\right) =$



$\left(\text{scaling by } \frac{1}{T}\right)$

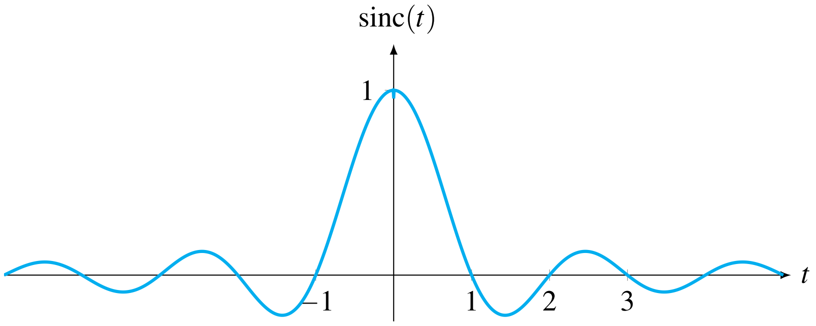
8. **Sinc and Sampling Functions**

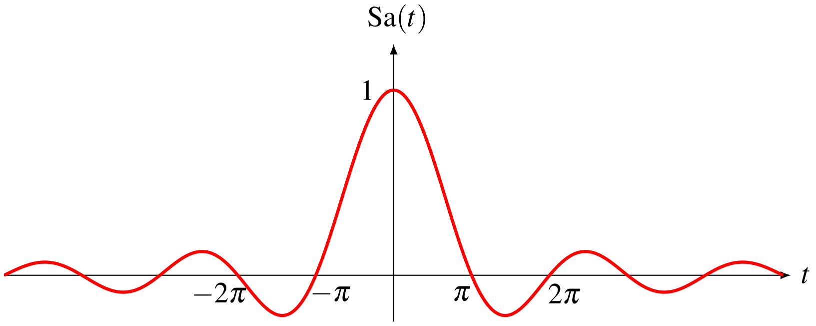
$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad \text{(note } \pi \text{ is explicitly present)} \tag{1.19}$$

R Textbook uses the definition $\operatorname{sinc}(t) = \frac{\sin t}{t}$. We will us a different name for that so we don't confuse them:

$$\operatorname{Sa}(t) = \frac{\sin(t)}{t} \tag{1.20}$$

We call $\operatorname{Sa}(t)$ the sampling function.





1.9 Plotting and Mathematical Representation of Signals Given By Straight-Line Segments

In this section we will look at plotting functions involving unit step functions ($u(t)$) and ramp functions ($r(t)$), as well as writing function in terms of unit step and ramp functions from a graph.

■ **Example 1.4** Plot $x(t) = 3u(t) + tu(t) - (t-1)u(t-1) - 5u(t-2)$.

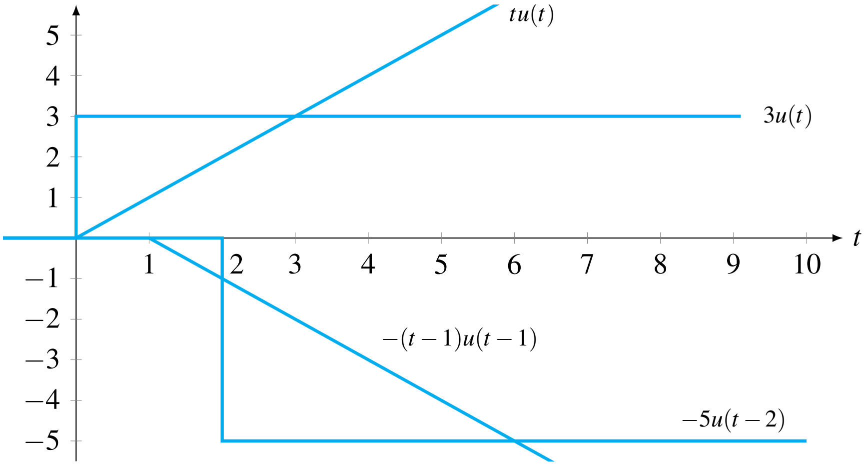
Individual terms:

$$3u(t) = \begin{cases} 3 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

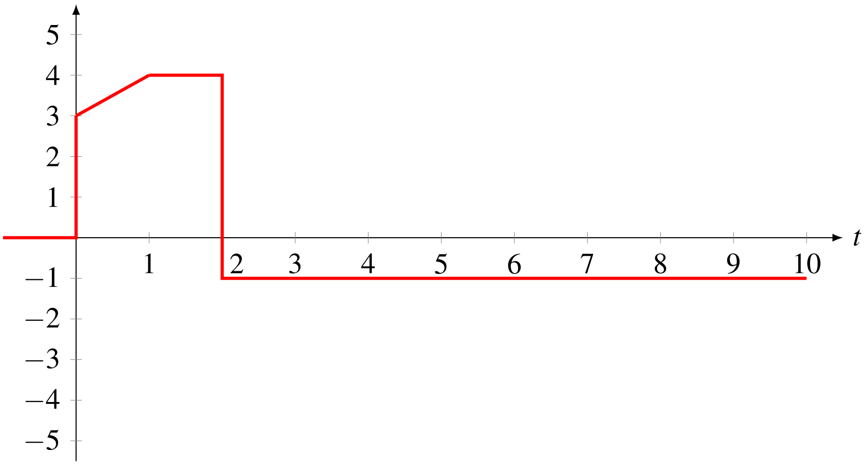
$$tu(t) = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$-(t-1)u(t-1) = \begin{cases} -(t-1) & \text{if } t-1 > 0 \\ 0 & \text{if } t-1 < 0 \end{cases} = \begin{cases} 1-t & \text{if } t > 1 \\ 0 & \text{if } t < 1 \end{cases}$$

$$-5u(t-2) = \begin{cases} -5 & \text{if } t > 2 \\ 0 & \text{if } t < 2 \end{cases}$$



⇓ combine them

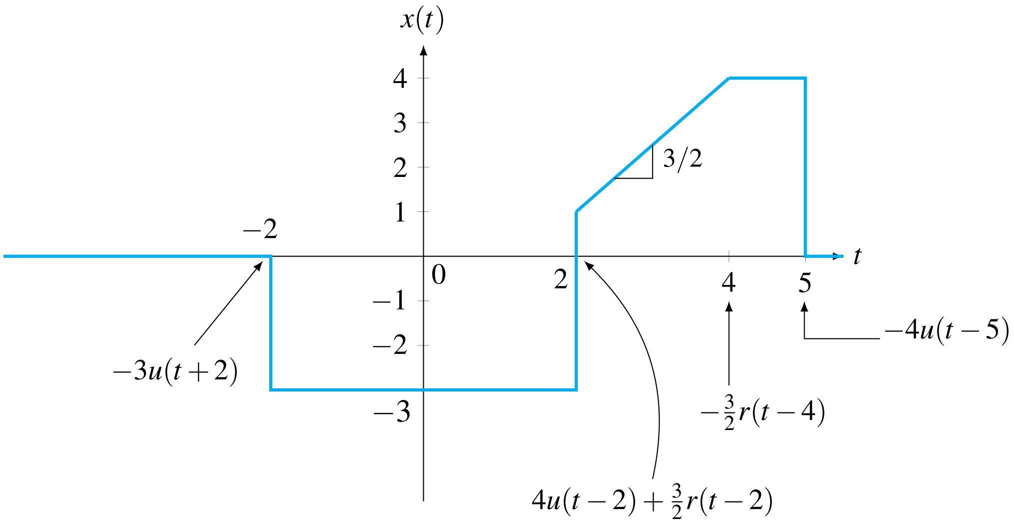


Faster Method:

- Look for points where there is a jump (e.g. $t = 0, 2$) or a change of slope (e.g. $t = 0, 1$).
- Calculate $x(t)$ at these points and connect the dots.

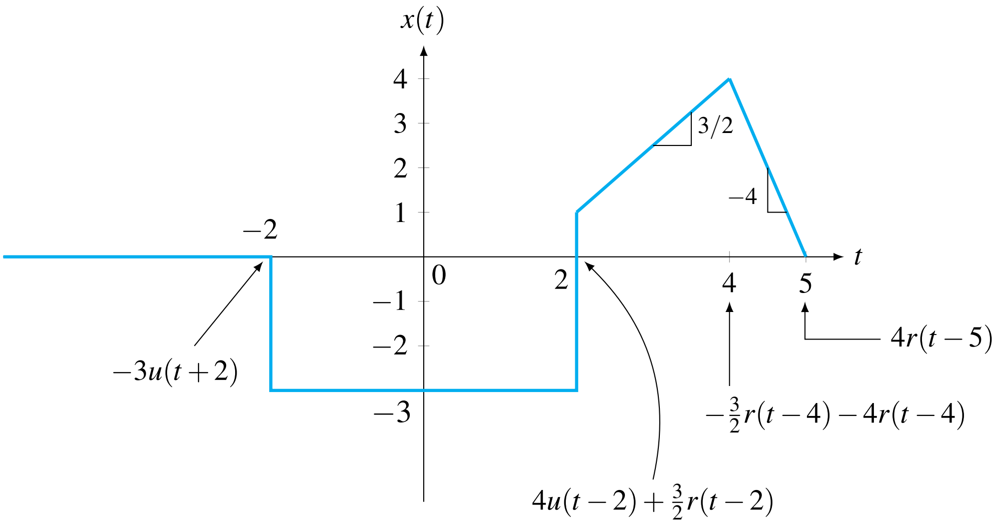
■

■ **Example 1.5** Represent the following signal:



$$x(t) = -3u(t+2) + 4u(t-2) + \frac{3}{2}r(t-2) - \frac{3}{2}r(t-4) - 4u(t-5)$$

Now suppose that the graph was changed to:



$$\begin{aligned} x(t) &= -3u(t+2) + 4u(t-2) + \frac{3}{2}r(t-2) - \frac{3}{2}r(t-4) - 4r(t-4) \\ &= -3u(t+2) + 4u(t-2) + \frac{3}{2}r(t-2) - \frac{11}{2}r(t-4) \end{aligned}$$

■

1.10 Important Engineering Signals: Unit Impulse or Dirac Delta Function $\delta(t)$

This is not a "typical" function (it is known as a "generalized function" by mathematicians). It is defined as the limit as

$$\delta(t) = \lim_{\varepsilon\to 0} f_\varepsilon(t) \tag{1.21}$$

of a sequence of conventional functions $f_\varepsilon(t)$ which have the following properties:

1.
$$\lim_{\varepsilon\to 0} f_\varepsilon(0) = +\infty$$

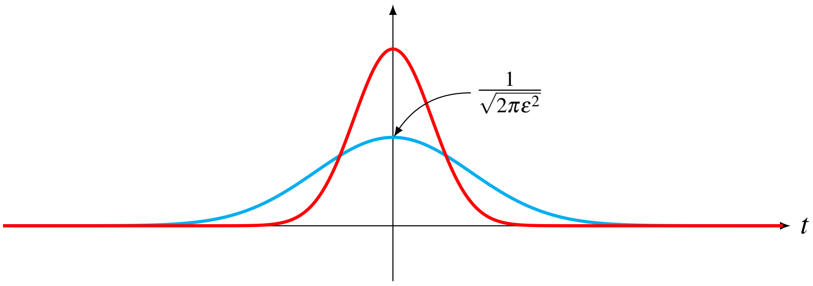
2.
$$\lim_{\varepsilon\to 0} f_\varepsilon(t) = 0,\quad \forall t \neq 0 \text{ (for all } t \neq 0)$$

3.
$$\int_{-\infty}^{\infty} f_\varepsilon(t)dt = 1,\quad \forall\varepsilon \text{ (for all } \varepsilon)$$

4.
$$f_\varepsilon(-t) = f_\varepsilon(t) \quad \text{(even function of } t)$$

5. (more technical assumption) $f_\varepsilon(t)$ is everywhere differentiable any number of times and $f_\varepsilon(t)$ and all its derivatives satisfy the condition
$$\lim_{t\to\infty} \frac{f_\varepsilon^{(n)}(t)}{t} = 0,\ n = 0, 1, 2, \ldots$$

■ **Example 1.6** $f_\varepsilon(t) = \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-\frac{t^2}{2\varepsilon^2}}$ (this is known as a Gaussian function).



1.
$$f_\varepsilon(0) = \frac{1}{\sqrt{2\pi\varepsilon^2}} \xrightarrow[\varepsilon\to 0]{} \infty$$

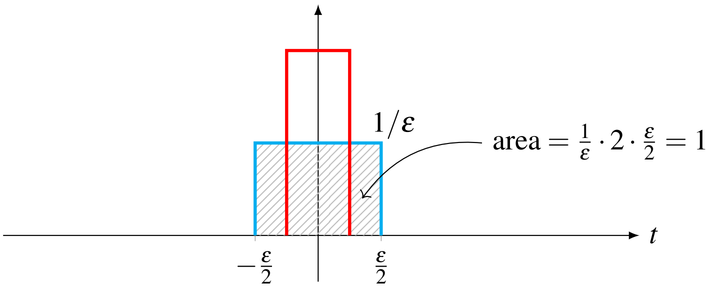
2.
$$f_\varepsilon(t) = \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-\frac{t^2}{2\varepsilon^2}} \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{(by L'Hospital Rule)}$$

3. Known fact from calculus
4. Even (obvious)
5. ✓

■

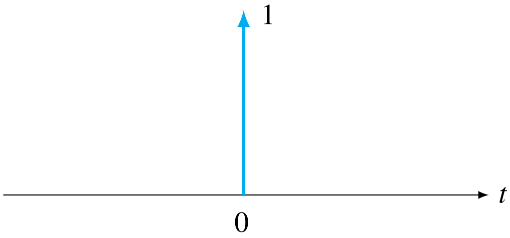
Most times, for purposes of exposition, when rigor is not important, we will dispose of property 5. In such case we can consider the sequence of functions

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right)$$

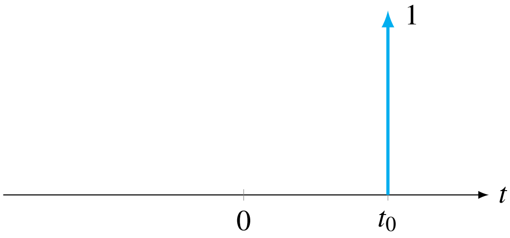


where all 4 properties can be readily verified.

Notation $\delta(t)$:



General case: $\delta(t - t_0)$



We say that the delta function "fires" at t_0.

1.10.1 Properties of the Delta Function

These properties follow from the definition of $\delta(t)$

1.

$$\delta(0) = +\infty$$

2.
$$\delta(t) = 0, \ \forall t \neq 0 \text{ (for all } t \neq 0)$$

3.
$$\int_{-\infty}^{\infty} \delta(t) = 1$$

4.
$$\delta(-t) = \delta(t)$$

5. ***Sifting Property:*** Let $x(t)$ be continuous at t_0. Then

$$\int_{t_1}^{t_2} x(t)\delta(t-t_0)dt = \begin{cases} x(t_0) & \text{if } t_0 \in (t_1,t_2) \\ 0 & \text{if } t_0 \notin [t_1,t_2] \\ \text{undefined} & \text{if } t_0 = t_1 \text{ or } t_0 = t_2 \end{cases}$$

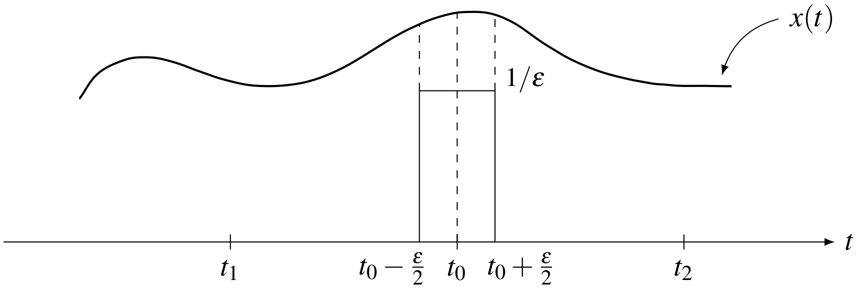
In other words, if the δ fires inside the (t_1,t_2) interval and $x(t)$ is continuous at this point, then the integral evaluates to $x(t_0)$.If $x(t)$ is discontinuous at t_0 then the integral is undefined.

■ Example 1.7

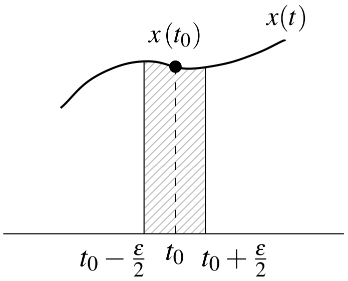
$$\int_{-1}^{1} e^{t^3}\delta(t)dt = e^{t^3}\Big|_{t=0} = 1$$
$$\int_{-2}^{-1} e^{t^3}\delta(t)dt = 0$$
$$\int_{-4}^{3} u(t)\delta(t)dt = \text{not defined since } u(t) \text{ is discontinuous at } t=0$$
$$\int_{-1}^{4} u(t)\delta(t-2)dt = u(t)|_{t=2} = 1$$
$$\int_{-\infty}^{t} x(\tau)\delta(\tau-t_0)d\tau = \begin{cases} x(t_0) & \text{if } t > t_0 \\ 0 & \text{if } t < t_0 \ \ (x(t) \text{ is continuous at } t_0) \\ \text{undefined} & \text{if } t = t_0 \end{cases}$$
$$\int_{0}^{\infty} \cos(t)e^{-2t}\delta(t+1)dt = 0$$

■

Intuition:

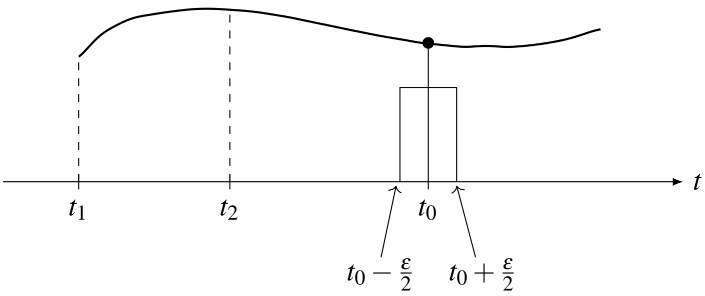


$$\begin{aligned}\int_{t_1}^{t_2} x(t)\delta\left(t-t_0\right)dt &= \int_{t_1}^{t_2} x(t)\left(\lim_{\varepsilon\to 0} f_\varepsilon(t)\right)dt\\ &\overset{!!!}{=} \lim_{\varepsilon\to 0}\int_{t_1}^{t_2} x(t)f_\varepsilon(t)dt\\ &= \lim_{\varepsilon\to 0}\int_{t_0-\frac{\varepsilon}{2}}^{t_0+\frac{\varepsilon}{2}} x(t)\frac{1}{\varepsilon}dt\end{aligned}$$



$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0-\frac{\varepsilon}{2}}^{t_0+\frac{\varepsilon}{2}} x(t)dt = x\left(t_0\right)$$

while

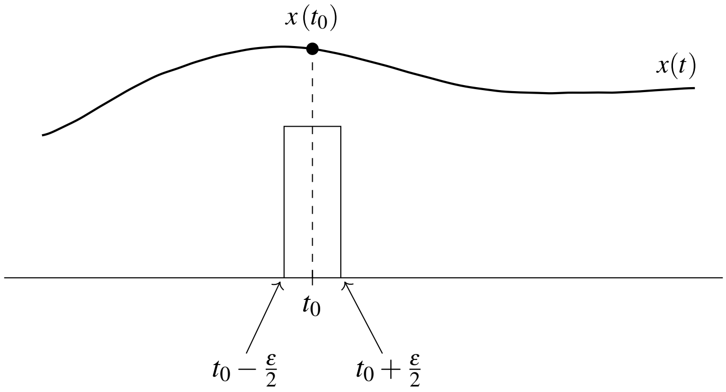


$$\begin{aligned}&\int_{t_1}^{t_2} x(t)\delta\left(t-t_0\right)dt\\ &\overset{!}{=} \lim_{\varepsilon\to 0}\int_{t_1}^{t_2} x(t)\cancelto{0}{f_\varepsilon(t)}\,dt = 0\end{aligned}$$

1.10.2 Sampling Property

If $x(t)$ is continuous at $t=t_0$ then

$$x(t)\delta\left(t-t_0\right) = x\left(t_0\right)\delta\left(t-t_0\right)$$



1.10.3 Scaling

Motivating example: $\int_{-\infty}^{\infty}\delta(5t)dt = ?$
A safe way to solve it is by using the following change of variables:

$$t' = 5t \implies dt' = 5dt$$

Hence,

$$\int_{-\infty}^{\infty}\delta(5t)dt = \int_{-\infty}^{\infty}\delta\left(t'\right)\frac{dt'}{5} = \frac{1}{5}\int_{-\infty}^{\infty}\delta\left(t'\right)dt' = \frac{1}{5}$$

Instead of having to do that in every integral involving scaled delta's, we will use the following identity:

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{1.22}$$

so

$$\int_{-\infty}^{\infty}\delta(at)dt = \int_{-\infty}^{\infty}\frac{1}{|a|}\delta(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty}\delta(t)dt = \frac{1}{|a|} \tag{1.23}$$

More generally,

$$\delta(at+b) = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right) \tag{1.24}$$

Proof. Consider the case $a>0$. Let

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\operatorname{rect}\left(\frac{t}{\varepsilon}\right)$$

Then,

$$\begin{aligned} f_\varepsilon(at+b) &= \frac{1}{\varepsilon}\operatorname{rect}\left(\frac{at+b}{\varepsilon}\right) \\ &= \frac{1}{\varepsilon}\operatorname{rect}\left(\frac{t+b/a}{\varepsilon/a}\right) \\ &= \frac{1}{a}\frac{1}{\varepsilon/a}\operatorname{rect}\left(\frac{t+b/a}{\varepsilon/a}\right) \end{aligned}$$

Now let $\varepsilon' = \varepsilon/a$.

$$\begin{aligned}\delta(at+b) = \lim_{\varepsilon\to 0} f_\varepsilon(at+b) &= \lim_{\varepsilon'\to 0} \frac{1}{a}\frac{1}{\varepsilon'}\,\mathrm{rect}\left(\frac{t+b/a}{\varepsilon'}\right)\\ &= \frac{1}{a}\lim_{\varepsilon'\to 0}\mathrm{rect}\left(\frac{t+b/a}{\varepsilon'}\right) = \frac{1}{a}\delta\left(t+\frac{b}{a}\right)\end{aligned}$$

Similarly for $a < 0$. ☐

- For the most part, the Dirac delta function will only appear **inside an integral**.
- In these cases it is safe to evaluate the integral using the sifting property.

From the sifting property we have

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} = u(t)$$

so $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$, and thus some authors proceed to differentiate the above equation to get

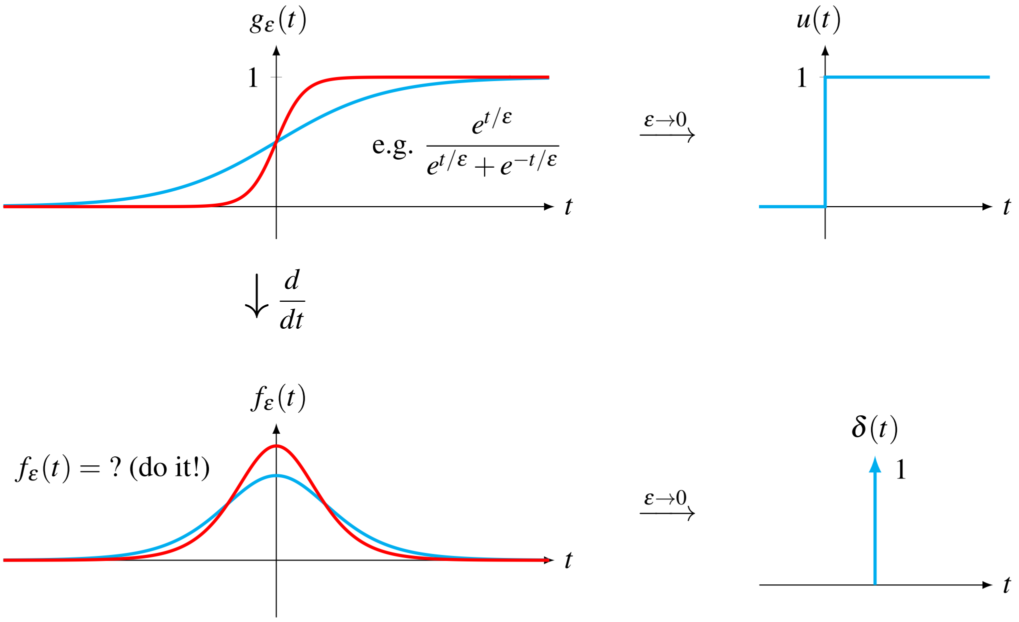
$$\frac{du(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{t} \delta(\tau)d\tau \underset{\substack{\uparrow\\ \text{from}\\ \text{fundamental}\\ \text{theorem of}\\ \text{calculus}}}{=} \delta(t)$$

and so they conclude (incorrectly!) that

$$\frac{du(t)}{dt} = \delta(t)$$

Note however that $\delta(t)$ is not a continuous function at $t = 0$ and $u(t)$ is not continuous (and thus not differentiable) at $t = 0$.

Under some circumstances it may be convenient to "formally" write $\frac{du(t)}{dt} = \delta(t)$, but care must be taken to explain the meaning of this statement.
e.g.



Derivative of Dirac Delta Function

Take $t_1 < 0 < t_2$

$$\begin{aligned}
\int_{t_1}^{t_2} f(t)\delta'(t)dt &= f(t)\delta(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2} f'(t)\delta(t)dt \\
&= f(t_2)\cancelto{0}{\delta(t_2)} - f(t_1)\cancelto{0}{\delta(t_1)} - f'(0) \\
&= f'(0) \;\text{ if } f'(0) \text{ is continuous at } 0
\end{aligned}$$

So, if $f(t)$ is continuously differentiable at t_0, then

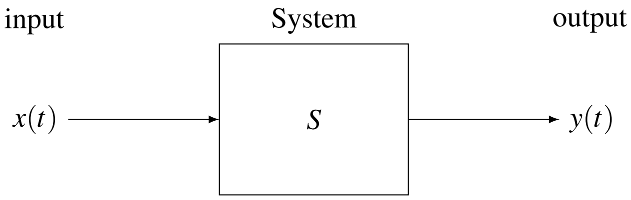
$$\int_{t_1}^{t_2} f(t)\delta'(t-t_0)\,dt = \begin{cases} -f'(t_0) & \text{if } t_0 \in (t_1, t_2) \\ 0 & \text{if } t_0 \notin [t_1, t_2] \\ \text{undefined} & \text{if } t_0 = t_1 \text{ or } t_0 = t_2 \end{cases}$$

2. Systems

2.1 Systems

A system is an entity that performs a function, i.e. operates on something (calling "input") and produces something else (called "output"). There are different types of systems: e.g. electrical, mechanical, biological, chemical, economic, social, etc.

In this class we will study systems that operate on a continuous-time analog signal $x(t)$ and produce a continuous-time analog signal $y(t)$.



Note that the system (in general) operates on the entire input x to produce y.

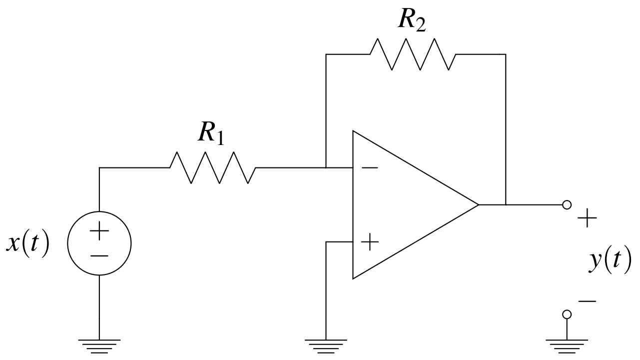
Notation:

$$y = \mathrm{S}[x]$$
$$y(t) = \mathrm{S}[x](t)$$

(compare with the misleading notation $y(t) = \mathrm{S}[x(t)]$).

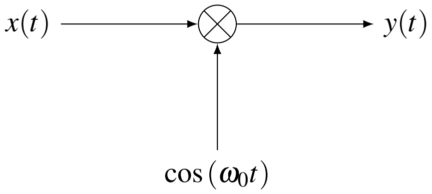
2.1.1 Examples of Systems

1. Constant gain (inverting amplifier).



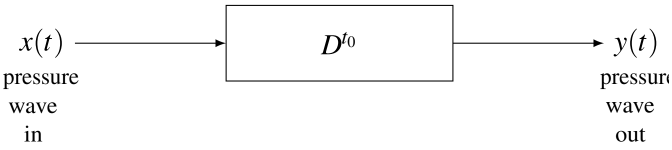
$$y(t) = -\frac{R_2}{R_1} x(t) \tag{2.1}$$

2. Modulation (or mixing)



$$y(t) = x(t)\cos(\omega_0 t) \tag{2.2}$$

3. Sound through a duct: pure time delay



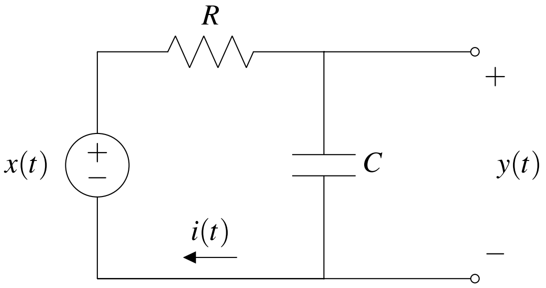
$$y(t) = x(t - t_0) \text{ where } t_0 = \frac{\text{length of pipe}}{\text{speed of sound}}$$

Notation:

$$y = D^{t_0}[x]$$

where D^{t_0} means delay by t_0.

4. RC circuit (no initial charge on the capacitor)

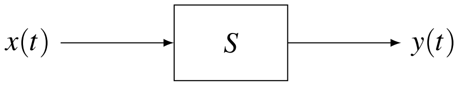


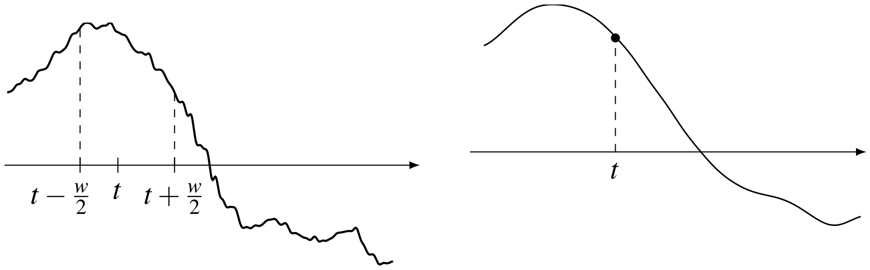
$$\left.\begin{aligned} [3]i(t) &= C\frac{dy(t)}{dt} && \text{(capacitor)} \\ y(t) &= x(t) - i(t)R && \text{(KVL)} \end{aligned}\right\} \implies y(t) = x(t) - RC\frac{dy(t)}{dt} \Leftrightarrow RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{2.3}$$

Equation (2.3) provides the input-output relationship of the RC circuit described by an ODE. One can show that

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} \frac{1}{RC} e^{-\frac{t-\tau}{RC}} x(\tau)d\tau \\ &= \mathrm{S}[x](t) \end{aligned} \tag{2.4}$$

5. Moving average filter





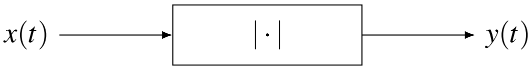
smooths out
input signal

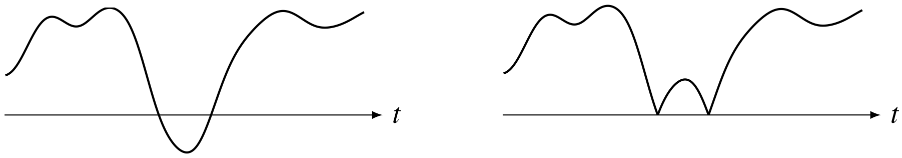
$$y(t) = \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x(\tau)d\tau \tag{2.5}$$

6. Pure integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

7. Rectifier





$$y(t) = |x(t)|$$

2.2 Classification of Systems

- memoryless (or static) vs with memory (or dynamical)
- causal vs non-causal
- linear vs non-linear
- time-invariant vs time-varying
- stable vs unstable

2.2.1 Memoryless

Definition 2.1 A system is said to be "***memoryless***" or "***static***" or instantaneous if the output at time t depends **ONLY** on the input at time t. Otherwise, it is said to be "with-memory" or "dynamical."

E.g., examples ①, ②, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ⑦ are memoryless. Memoryless systems are very simple mathematically.

2.2.2 Causal

Definition 2.2 A system is said to be ***causal*** (or ***non-anticipatory***) if the output at any time t does not depend on the input at times $\tau > t$.

E.g., examples ①, ②, ③, ④, ~~5~~, ⑥, ⑦ are causal (for #4, see close form solution).

> R Any physical system must be causal!

2.2.3 Linear

Consider a system S:

- The response of the system to $x_1(t)$ is $y_1(t)$

$$\text{notation: } x_1(t) \longrightarrow \boxed{S} \longrightarrow y_1(t)$$

$$\text{more formally: } y_1(t) = S[x_1](t)$$

- The response of the system to $x_2(t)$ is $y_2(t)$:

$$x_2(t) \longrightarrow \boxed{S} \longrightarrow y_2(t)$$

$$y_2(t) = S[x_2](t)$$

- Consider two constant numbers α_1, α_2

Motivating Question: What is the response of S when the input is the linear combination

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \boxed{S} \longrightarrow ?$$

Definition 2.3 A system S is called ***linear*** if for any inputs $x_1(t)$, $x_2(t)$ and any numbers α_1, α_2, the output of the system to the input

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

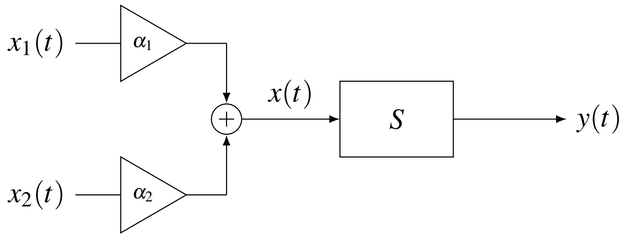
is

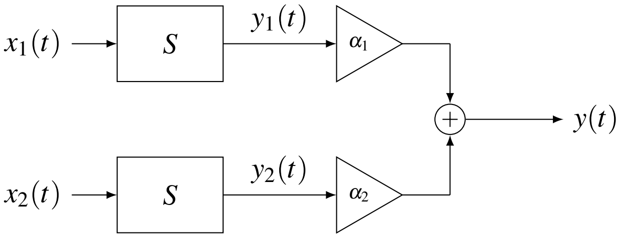
$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) \tag{2.6}$$

Formally:

$$S[\alpha_1 x_1 + \alpha_2 x_2](t) = \alpha_1 S[x_1](t) + \alpha_2 S[x_2](t), \text{ (for all } x_1, x_2, \alpha_1, \alpha_2, t) \tag{2.7}$$

Pictorially:





Which of the 7 examples studied are linear?

■ **Example 2.1** Determine whether Ex. 1 of the 7 examples studied is linear or nonlinear.

$$\begin{aligned}
S[x_1](t) &= -\frac{R_2}{R_1}x_1(t) \\
S[x_2](t) &= -\frac{R_2}{R_1}x_2(t) \\
S[\alpha_1 x_1 + \alpha_2 x_2](t) &= -\frac{R_2}{R_1}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\
&= \alpha_1 \left[-\frac{R_2}{R_1}x_1(t)\right] + \alpha_2 \left[-\frac{R_2}{R_1}x_2(t)\right] \\
&= \alpha_1 S[x_1](t) + \alpha_2 S[x_2](t)
\end{aligned}$$

so it is linear. ✓ ■

■ **Example 2.2** Determine whether Ex. 5, the moving average filter, is linear or nonlinear.

$$\begin{aligned}
S[x_1](t) &= \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x_1(\tau)d\tau \\
S[x_2](t) &= \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x_2(\tau)d\tau \\
S[\alpha_1 x_1 + \alpha_2 x_2](t) &= \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]d\tau \\
&= \alpha_1 \left(\frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x_1(\tau)d\tau\right) + \alpha_2 \left(\frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x_2(\tau)d\tau\right) \\
&= \alpha_1 S[x_1](t) + \alpha_2 S[x_2](t)
\end{aligned}$$

so it is linear. ✓ ■

■ **Example 2.3** Determine whether Ex. 7, the rectifier, is linear or nonlinear.

$$\begin{aligned}
S[x_1](t) &= |x_1(t)| \\
S[x_2](t) &= |x_2(t)| \\
S[\alpha_1 x_1 + \alpha_2 x_2](t) &= |\alpha_1 x_1(t) + \alpha_2 x_2(t)| \\
&\neq \alpha_1 |x_1(t)| + \alpha_2 |x_2(t)|
\end{aligned}$$

so this system is nonlinear. ■

Of the seven examples we considered, 1 – 6 are all linear, but 7 is non-linear. Do it!

disable ①, ②, ③, ④, ⑤, ⑥, ~~7~~

R

- The property $S[\alpha x](t) = \alpha S[x](t)$ is called "homogeneity."
- The property $S[x_1 + x_2](t) = S[x_1](t) + S[x_2](t)$ is called "superposition."

So Linearity ⇔ homogeneity AND superposition. Show it!

Proof.

1. Linearity $\implies$ homogeneity:
   Set $\alpha_2 = 0$. Then from linearity,
   $$\begin{aligned} &S[\alpha_1 x_1 + 0x_2](t) = \alpha_1 S[x_1](t) + 0S[x_2](t) \\ \iff\ &S[\alpha_1 x_1](t) = \alpha_1 S[x_1](t) \ \ \text{(homogeneity)} \end{aligned}$$
2. Linearity $\implies$ superposition:
   Set $\alpha_1 = \alpha_2 = 1$.
3. $\Leftarrow$? Do it!

$\square$

Based on the above, one can easily show that the response of a linear system when the input is $x(t) = 0$ must be $y(t) = 0$.

Proof. Indeed, Linearity $\implies$ Homogeneity, i.e.

$$S[ax](t) = aS[x](t)$$

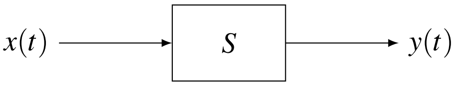
and substituting $a = 0$, we get

$$\begin{aligned} &S[0x](t) = 0S[x](t) \\ \iff\ &S[0](t) = 0,\ \forall t \text{ (for all } t) \end{aligned}$$

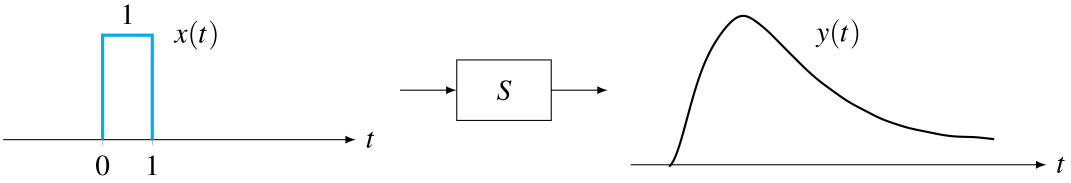
Note that the 0 in $S[0]$ is the zero signal, i.e. $x(t) = 0,\ \forall t$, whereas the 0 after the equals sign represents the number zero. $\square$

2.2.4 Time Invariant

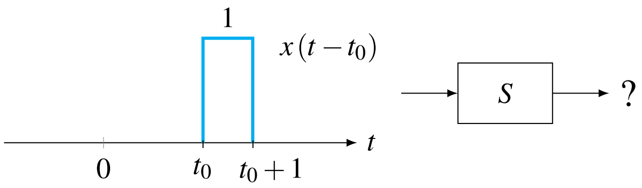
Consider a system S with input $x(t)$ and output $y(t)$:



Motivating question:



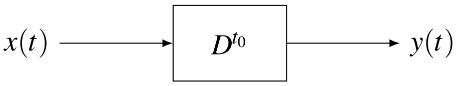
Suppose you input a delayed version of $x(t)$ to the system, i.e. $x(t-t_0)$. Will you get a delayed version of $y(t)$, i.e. $y(t-t_0)$?



Answer: NOT ALWAYS.

Definition 2.4 A system is said to be ***time-invariant*** if for any input $x(t)$ and any delay t_0, the output of the system to the input $x(t-t_0)$ is $y(t-t_0)$.

Formally: recall that a pure delay system



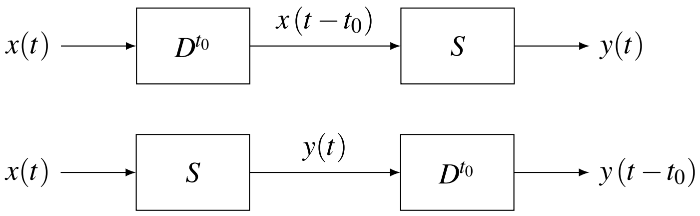
is denoted by D^{t_0}, i.e.

$$D^{t_0}[x](t) = x(t-t_0) \tag{2.8}$$

Time-invariant:

$$S\left[D^{t_0}[x]\right](t) = S[x](t-t_0) = D^{t_0}[S[x]](t) \tag{2.9}$$

Pictorially:



In other words, S and D^{t_0} commute.

Which of the seven examples studied are time invariant?

■ **Example 2.4** Determine whether Ex. 6, the pure integrator, is time invariant or time variant.

$$S[x](t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$S[x](t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

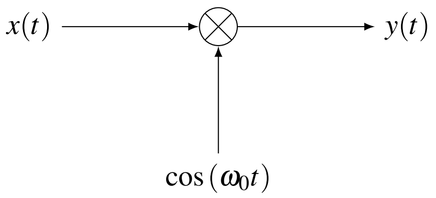
$$S\left[D^{t_0}[x]\right](t) = \int_{-\infty}^{t} x(\tau-t_0)\,d\tau$$

Using the change of variables $s = \tau - t_0 \implies ds = d\tau$,

$$S\left[D^{t_0}[x]\right](t) = \int_{-\infty}^{t-t_0} x(s)ds$$

Since $S[x](t-t_0) = S[D^{t_0}[x]](t)$, S is **time invariant**. ■

■ **Example 2.5** Determine whether Ex. 2, the modulator, is time invariant or time variant.



$$\begin{aligned} S[x](t) &= x(t)\cos(\omega_0 t) \\ S[x](t-t_0) &= x(t-t_0)\cos[\omega_0(t-t_0)] \\ S\left[D^{t_0}[x]\right](t) &= x(t-t_0)\cos(\omega_0 t) \end{aligned}$$

Since $S[x](t-t_0) \neq S[D^{t_0}[x]]$, this system is **not** time invariant, or it is a **time-varying** system. ■

Of the seven systems presented, 1 and 3–7 are all time-invariant, but 2 is time-varying. Do it!

①, ~~2~~, ③, ④, ⑤, ⑥, ⑦

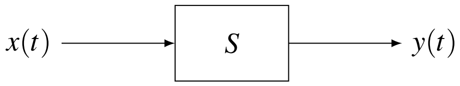
2.2.5 Stability

Recall that a signal $x(t)$ is called ***bounded*** if there exists some constant M such that

$$|x(t)| \leq M, \ \forall t \text{ (for all } t) \tag{2.10}$$

Bounded Input-Bounded Output (BIBO) Stability

Definition 2.5 A system is said to be BIBO stable if for every bounded input, the output is also bounded.



$$\text{if } |x(t)| < M_1 \text{ then } |y(t)| < M_2$$

■ **Example 2.6** Determine whether Ex. 2, the modulator, is BIBO stable.

$$\begin{aligned} y(t) &= x(t)\cos(\omega_0 t) \\ \implies |y(t)| &= |x(t)\cos(\omega_0 t)| \leq |x(t)| \end{aligned}$$

So if $|x(t)| \leq M_1$, then $|y(t)| \leq |x(t)| \leq M_1$, so it is BIBO stable. ■

■ **Example 2.7** Determine whether Ex. 5, the moving average filter, is BIBO stable.

$$\begin{aligned} y(t) &= \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x(\tau)d\tau \\ |y(t)| &= \left|\frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x(\tau)d\tau\right| \leq \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} |x(\tau)|d\tau \end{aligned}$$

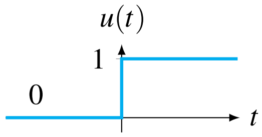
So if $|x(\tau)| \leq M_1$,

$$|y(t)| \leq \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} |x(\tau)|d\tau \leq \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} M_1 d\tau = M_1 \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} d\tau = M_1$$

so it is BIBO stable. ■

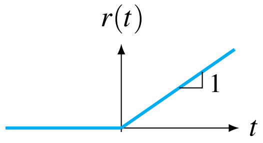
■ **Example 2.8** Determine whether Ex. 6, the integrator, is BIBO stable.
Take

$$x(t) = u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$



$|x(t)| \leq 1$, so it is bounded. However,

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$



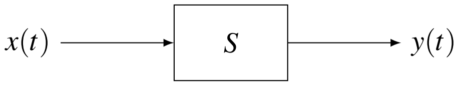
and $y(t)$ is an unbounded signal, so the integrator is NOT BIBO stable. ■

Of the 7 example systems:

①, ②, ③, ④, ⑤, ~~⑥~~, ⑦

2.3 A Common Class of Systems: Systems Described by LCCDEs

A common class of systems are those described by linear, constant-coefficient differential equations (LCCDEs).



where

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t)$$
$$= b_0 \frac{d^M x(t)}{dt^M} + \cdots + b_{M-1} \frac{dx(t)}{dt} + b_M x(t) \tag{2.11}$$

where $a_1, a_2, \ldots, a_N$ and $b_0, b_1, \ldots, b_M$ are real coefficients and $N > M$.

- If we want to find the output of the system for all $t > t_0$, we need N initial conditions $y(t_0)$, $y'(t_0), \ldots, y^{(n-1)}(t_0)$[1]
- N is the "***order***" or "***dimension***" of the system.
- As it turns out, the overall response of this system is the sum of two parts:

$$y(t) = y_{ZIR}(t) + y_{ZSR}(t), \; t \geq t_0 \tag{2.12}$$

where $y_{ZIR}(t)$ is the response of the system when $x(t) = 0$ (Zero Input Response) and $y_{ZSR}(t)$ is the response of the system when the initial conditions (state) are set to zero (Zero State Response).

---

[1] When $x(t)$ or its derivatives contain δ functions, we need to know initial conditions at t_0^- because y and its derivatives can change instantaneously from t_0^- to t_0^+.

Special Case: Consider the first order system described by the LCCDE

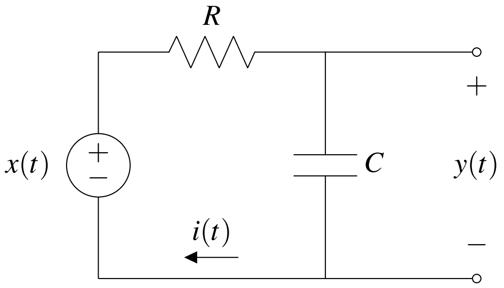
$$\frac{dy(t)}{dt} + \alpha y(t) = bx(t),\ y\left(t_0^-\right) = y_o,\ t \geq t_0$$

(also use notation $\dot{y}(t)$ for $\frac{dy(t)}{dt}$).
Solution:

$$y(t) = y_{ZIR}(t) + y_{ZSR}(t) = y_0 e^{-a(t-t_0)} + \int_{t_0}^{t} be^{-a(t-\tau)} x(\tau) d\tau,\ t \geq t_0$$

Note that in the above solution, $y_{ZIR}(t) = y_0 e^{-a(t-t_0)}$ (the first term), and $y_{ZSR} = \int_{t_0}^{t} be^{-a(t-\tau)} x(\tau) d\tau$ (the second term).

■ **Example 2.9** Consider the RC circuit shown below:

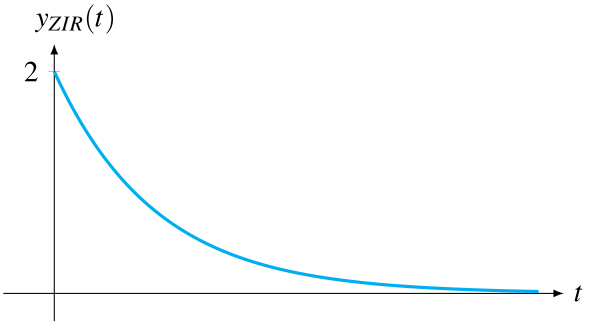


$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

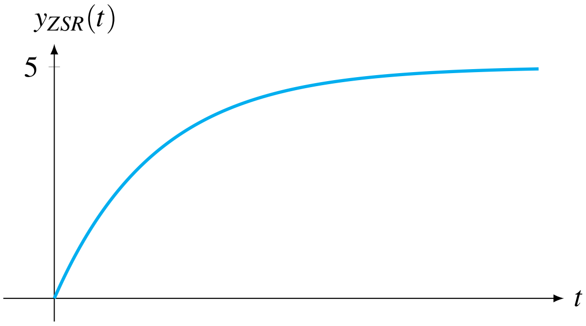
Specific values:

> $t_0 = 0$
> $y(0) = y_0 = 2$
> $x(t) = 5u(t) = \begin{cases} 5 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$
> $RC = \frac{1}{100}$ (sec)

$$y_{ZIR}(t) = 2e^{-100t},\ t \geq 0$$

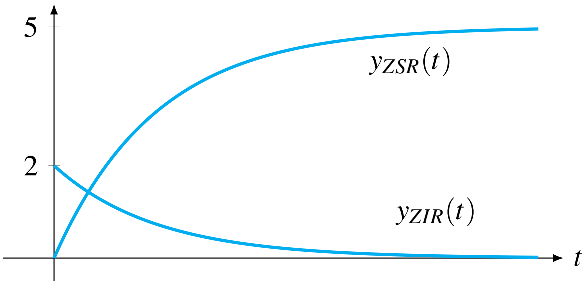


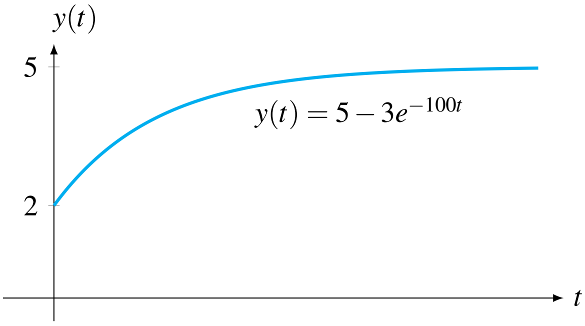
$$
\begin{aligned}
y_{ZSR}(t) &= \int_0^t 100e^{-100(t-\tau)}5d\tau = 500e^{-100t}\int_0^t e^{100\tau}d\tau \\
&= 500e^{-100t}\frac{e^{100\tau}}{100}\Big|_0^t = 500e^{-100t}\frac{e^{100t}-1}{100} \\
&= 5\left(1-e^{-100t}\right),\ t \geq 0
\end{aligned}
$$



Overall:

$$y(t) = 2e^{-100t} + 5\left(1-e^{-100t}\right) = 5 - 3e^{-100t},\ t \geq 0$$





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3. Linear, Time-Invariant (LTI) Systems

For the rest of this course (and unless otherwise noted) we will be studying LTI systems. Why?

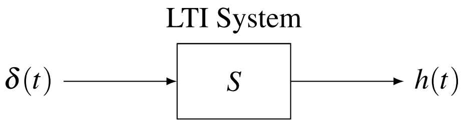
1. Good (approximate) models for many physical systems.
2. Mathematically very tractable $\implies$ analysis yields insight.

Our goal for now:

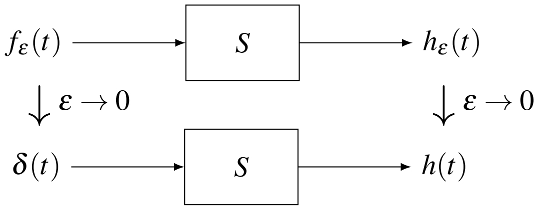
- Understand the general way of representing all LTI systems.
- Evaluate output from input using **convolution**.

3.1 Impulse Response (IR)

Definition 3.1 The ***impulse response*** $h(t)$ of an LTI system is the response of that system when the input is a Dirac delta function.



Since $\delta(t)$ is not a proper function, we interpret the IR as a limit:



$$h(t) = S[\delta](t) \tag{3.1}$$

The IR of an LTI system is very important. As it turns out, if we know $h(t)$ we can evaluate the output of the system to any input $x(t)$.

Proposition 3.1 (another word for a basic result that we will prove) For any LTI system with IR $h(t)$, the output $y(t)$ to an input $x(t)$ is given by the convolution integral, i.e.

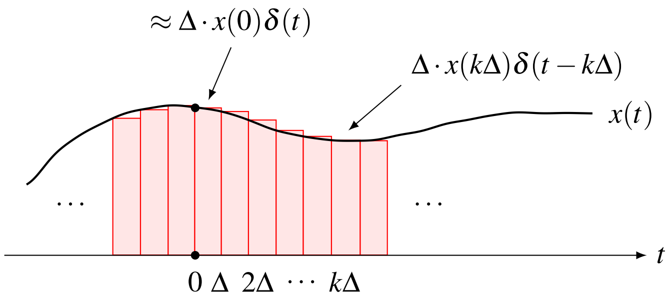
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{at any } t \text{ where } x \text{ is continuous} \tag{3.2}$$

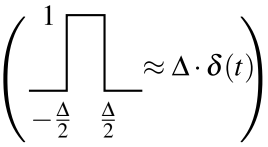
Notation:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= (x*h)(t) \quad \leftarrow \text{ better notation} \end{aligned}$$

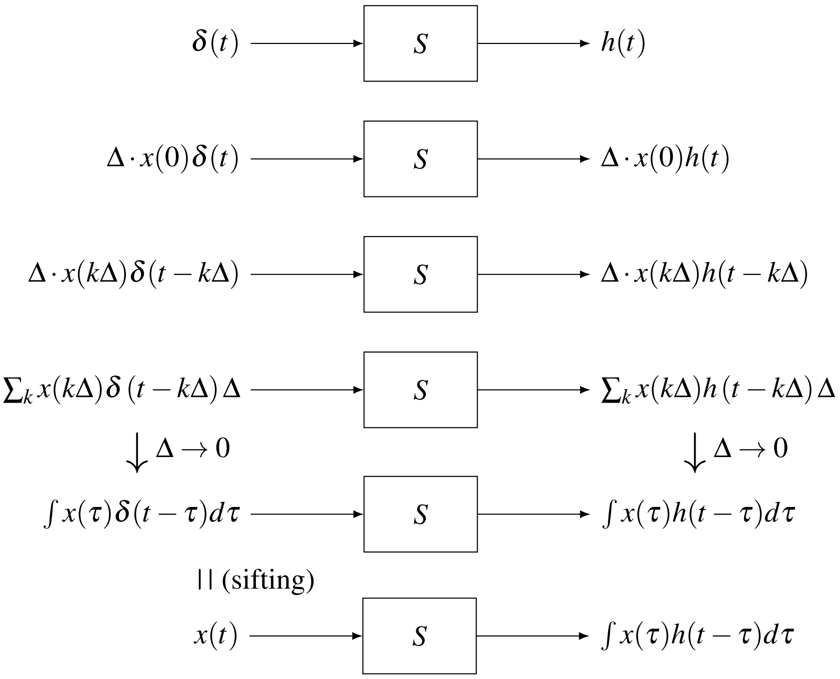
Why is $(x*h)(t)$ better notation?
Observe: $y(t)$ depends on all values of x and h, not just $x(t)$, $h(t)$.

3.1.1 Simple Explanation

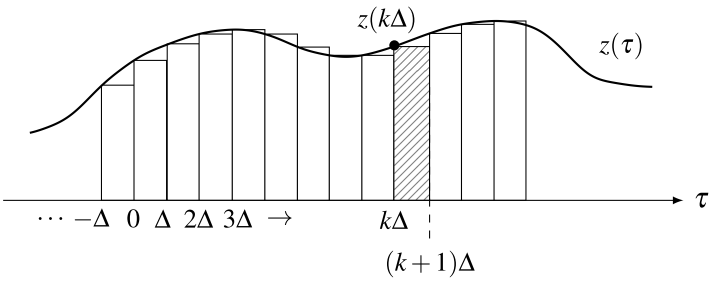




We know:

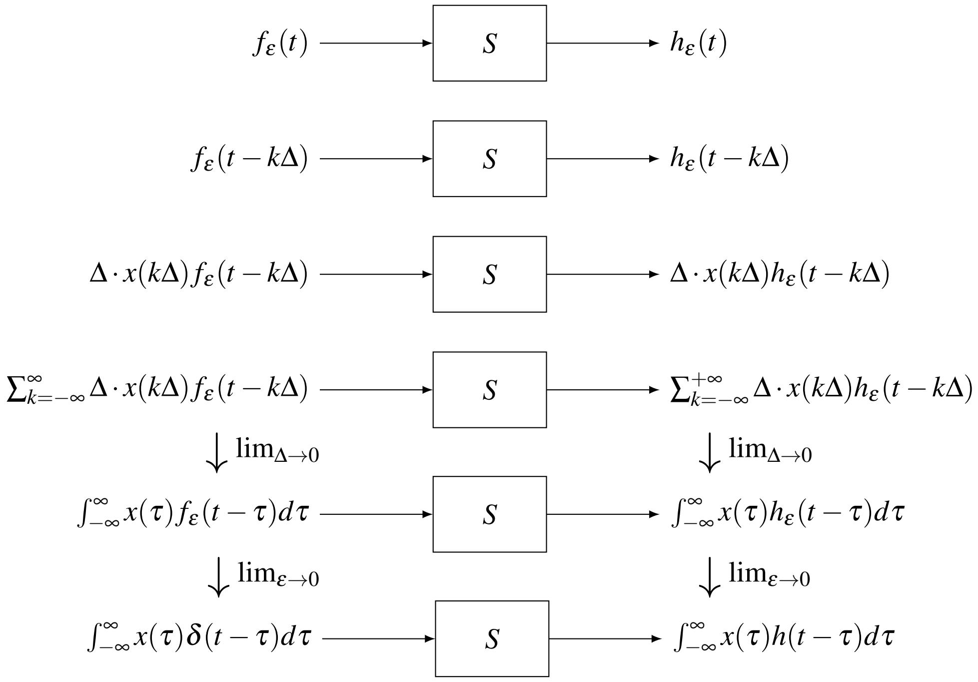


Proof. Recall the basic idea of the Riemann integral:

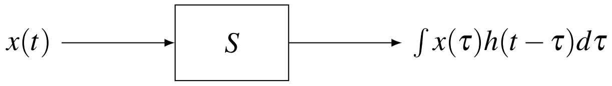


$$\int_{-\infty}^{\infty} z(\tau)d\tau = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} z(k\Delta)\cdot\Delta$$

Also recall the interpretation of IR:



By the sifting property, $\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$, so



□

■ **Example 3.1** Integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

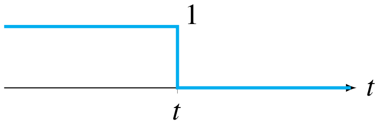
> So we already know how to evaluate $y(t)$ from any $x(t)$, but let's verify it.

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

Consider an arbitrary input $x(t)$:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau \end{aligned}$$

$$u(t-\tau) = \begin{cases} 1 & t > \tau \\ 0 & t < \tau \end{cases}$$



$$= \int_{-\infty}^{t} x(\tau) d\tau \checkmark$$

■

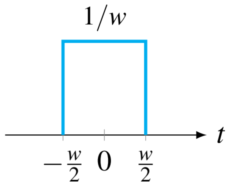
■ Example 3.2 Moving average filter:

$$y(t) = \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x(\tau)d\tau$$

> Similarly here we know an explicit expression for how $y(t)$ is evaluated from $x(t)$.

$$h(t) = \frac{1}{w}\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} \delta(\tau)d\tau \underset{\substack{\uparrow \\ \text{sifting property}}}{=} \begin{cases} \frac{1}{w} & 0 \in \left(t-\frac{w}{2}, t+\frac{w}{2}\right) \\ 0 & \text{else} \end{cases}$$

$=$



$$\begin{gathered} t-\tfrac{w}{2} < 0 < t+\tfrac{w}{2} \\ t < \tfrac{w}{2} \text{ or } t > -\tfrac{w}{2} \\ -\tfrac{w}{2} < t < \tfrac{w}{2} \end{gathered}$$

Consider an arbitrary input $x(t)$.

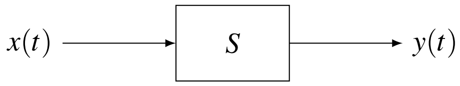
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\frac{w}{2}}^{t+\frac{w}{2}} x(\tau)\frac{1}{w}d\tau \ \checkmark$$

$$\uparrow$$

$$\begin{aligned} -\frac{w}{2} < t-\tau < \frac{w}{2} &\iff -\frac{w}{2} - t < -\tau < \frac{w}{2} - t \\ &\iff \frac{w}{2} + t > \tau > t - \frac{w}{2} \end{aligned}$$

■

■ Example 3.3



$$\begin{aligned} &y'(t) + ay(t) = bx(t) \\ &\text{zero initial conditions } y(-\infty) = 0 \end{aligned}$$

Q: Is this an LTI system?[1]
A: Yes.

$$x_1(t) \to y_1(t) \qquad y_1'(t) + ay_1(t) = bx_1(t) \tag{1}$$

$$x_2(t) \to y_2(t) \qquad y_2'(t) + ay_2(t) = bx_2(t) \tag{2}$$

$$\alpha_1 \cdot (1) + \alpha_2 \cdot (2) \iff \alpha_1\,[\text{LHS}_1] + \alpha_2\,[\text{LHS}_2] = \alpha_1\,[\text{RHS}_1] + \alpha_2\,[\text{RHS}_2]$$

$$\iff \frac{d}{dt}\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] + a\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] = b\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right]$$

$$\iff y'(t) + ay(t) = bx(t)$$

---

[1] All systems described by LCCDE with zero initial conditions at $t_0 = -\infty$ are LTI systems! (or else the ZSR is LTI)

so it is linear.

Similarly for TI: What is the output when the input is $x(t-t_0)$? Check if $y(t-t_0)$ satisfies the differential equation.

$$y'(t-t_0)+ay(t-t_0)\overset{?}{=}bx(t-t_0)$$

$$\uparrow$$

YES (change of variable $t-t_0=s$)

What is $h(t)$? Set $x(t)=\delta(t)$.

$$\begin{aligned} h'(t)+ah(t) &= b\delta(t) \\ h'(-\infty) &= 0 \end{aligned} \tag{1}$$

for $t<0$, $h(t)=0$ satisfies (1)
for $t>0$, $h(t)=Ce^{-at}$ also satisfies (1) (homogeneous equation)
Since $C(-a)e^{-at}+aCe^{-at}=0$ ✓
So $h(t)=Ce^{-at}u(t)$ is our guess

Check: Substitute our guess for $h(t)$ into (1)

$$\begin{aligned} C(-a)\cancel{e^{-at}u(t)}+Ce^{-at}\delta(t)+aC\cancel{e^{-at}u(t)} &= b\delta(t) \\ \iff Ce^{-at}\delta(t) &= b\delta(t) \\ \left(\begin{matrix}\text{sampling}\\ \iff \\ \text{property}\end{matrix}\right) \quad C\delta(t) &= b\delta(t) \\ \implies C=b \end{aligned}$$

so:

$$h(t)=be^{-at}u(t)$$

For any input $x(t)$, the output should be:

$$y(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau=\int_{-\infty}^{\infty}be^{-a(t-\tau)}u(t-\tau)x(\tau)d\tau$$

Since in order for $u(t-\tau)$ to be not equal to zero, $t-\tau>0 \iff t>\tau$, we get

$$y(t)=\int_{-\infty}^{t}be^{-a(t-\tau)}x(\tau)d\tau$$

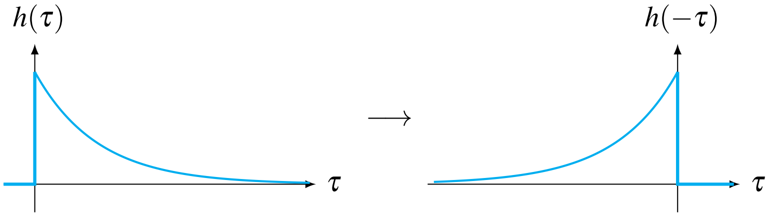
Compare the above result with the RC system #4. ■

3.2 Graphical Evaluation of Convolution

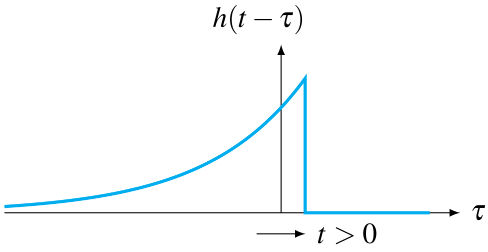
Suppose we are given two signals $h(t)$ and $x(t))$ and want to evaluate their convolution $y(t)=(x*h)(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$.

Procedure:

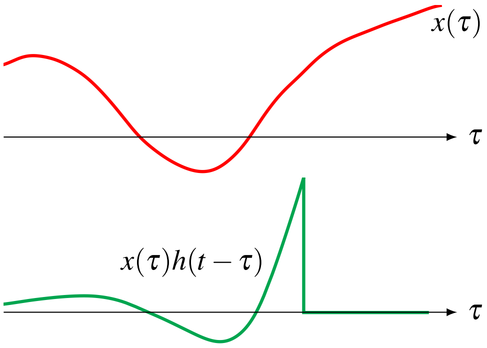
1. Flip $h(\tau) \rightarrow h(-\tau)$



2. Shift $h(-\tau)$ by $t \rightarrow h[-(\tau - t)] = h(t - \tau)$



3. Multiply $x(\tau)h(t-\tau)$



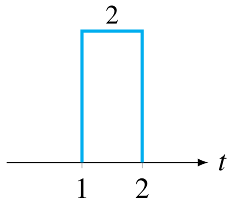
4. Integrate with respect to τ over $-\infty$ to $+\infty$.

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = y(t)$$

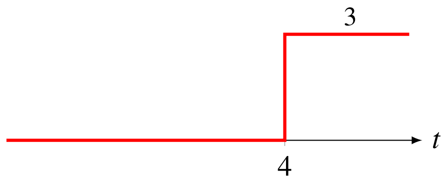
This process needs to be repeated for every value of t. But as we will see in the following examples, we can cover all values of t with a few cases.

■ Example 3.4

$$h(t) = \begin{cases} 2 & 1 < t < 2 \\ 0 & \text{else} \end{cases} =$$

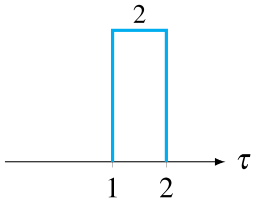


$$x(t) = 3u(t-4) = \begin{cases} 3 & t > 4 \\ 0 & t < 4 \end{cases} =$$



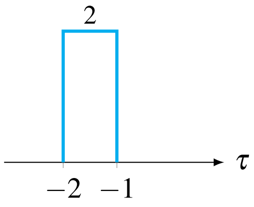
Evaluate $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$, $\forall t$ (for all t).

$$h(\tau) =$$



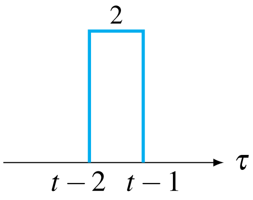
1. Reflect:

$$h(-\tau) =$$



2. Shift by t:

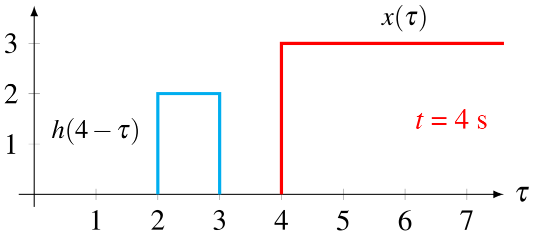
$$h(t-\tau) =$$



3. Multiply with $x(t)$ and integrate with respect to τ.

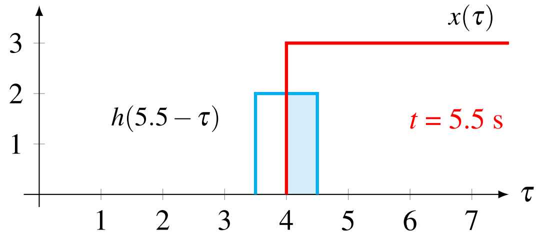
Three cases:

- *Case 1:* $t - 1 < 4 \iff t < 5$:



Since there is no overlap, $x(\tau)h(t-\tau)=0$, $\forall\tau$ (for every τ), so $\boxed{y(t)=0,\ \ t<5.}$

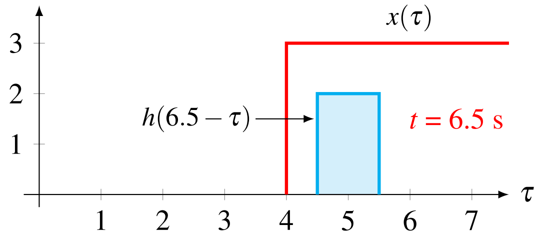
- *Case 2:* $t-2<4<t-1 \iff t<6 \text{ or } t>5 \iff 5<t<6$:



There is partial overlap in the range $\tau\in(4,t-1)$. Therefore,

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{4}^{t-1}6d\tau=6\tau|_4^{t-1}=6(t-1)-6(4)$$
$$=\boxed{\text{6t - 30, } 5<t<6}$$

- *Case 3:* $t-2>4 \iff t>6$:

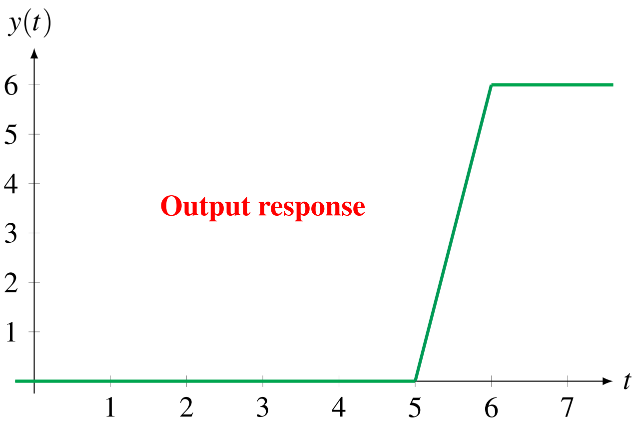


There is overlap for $\tau\in(t-2,t-1)$. Hence,

$$y(t)=\int_{t-2}^{t-1}6d\tau=6\tau|_{t-2}^{t-1}=6(t-1)-6(t-2)=\boxed{\text{6, t} > 6}$$

Overall:

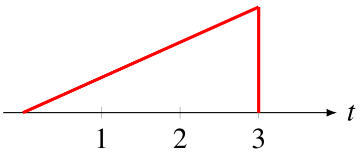
$$y(t)=\begin{cases}0 & \text{if } t<5\\ 6t-30 & \text{if } 5<t<6=6r(t-5)-6r(t-6)\\ 6 & \text{if } 6<t\end{cases}$$



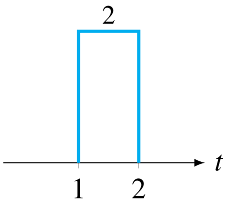
■

■ Example 3.5

$x(t) =$

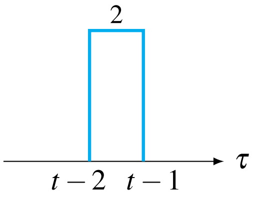


$h(t) =$



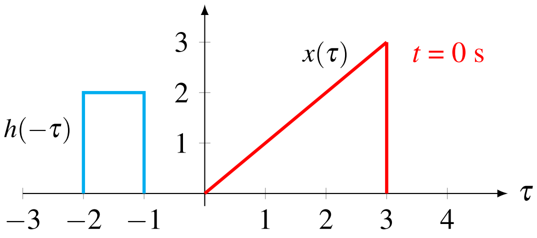
1) + 2):

$h(t-\tau) =$

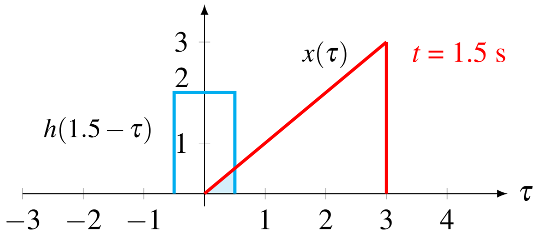


3) + 4): Five Cases

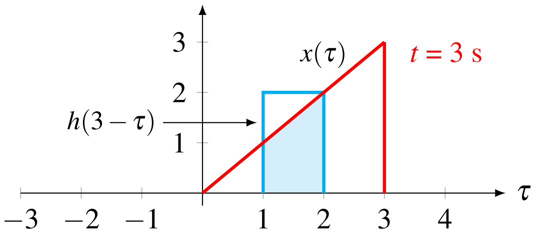
- *Case 1:* $t-1<0 \iff t<1 \longrightarrow y(t)=0$



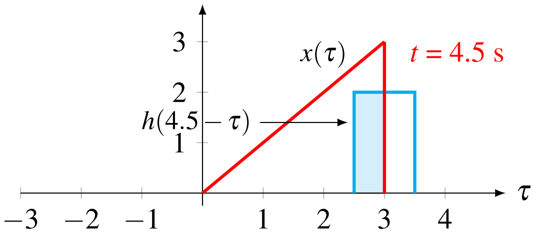
- *Case 2:* $t-2<0<t-1 \iff 1<t<2 \longrightarrow y(t)=\int_0^{t-1}\tau\cdot 2d\tau=\tau^2\big|_0^{t-1}=(t-1)^2$



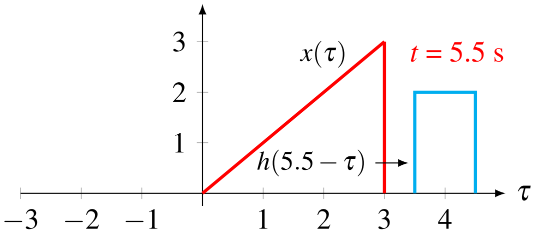
- *Case 3:* $t-2>0\ \&\ t-1<3 \iff 2<t<4 \longrightarrow$
  $y(t)=\int_{t-2}^{t-1}2\tau d\tau=\tau^2\big|_{t-2}^{t-1}=(t-1)^2-(t-2)^2=2t-3$



- *Case 4:* $t-2 < 3 < t-1 \iff 4 < t < 5 \longrightarrow$
  $y(t) = \int_{t-2}^{3} 2\tau d\tau = \tau^2\big|_{t-2}^{3} = 3^3 - (t-2)^2 = 5 + 4t - t^2$

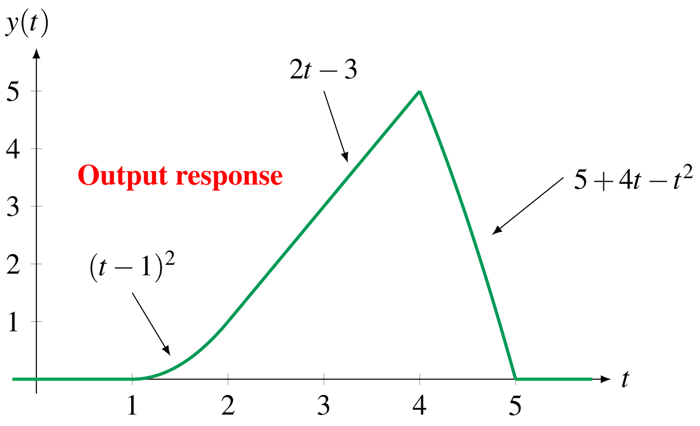


- *Case 5:* $t-2 > 3 \iff t > 5 \rightarrow y(t) = 0$



Overall:

$$y(t) = \begin{cases} 0 & \text{if } t < 1 \\ (t-1)^2 & \text{if } 1 < t < 2 \\ 2t-3 & \text{if } 2 < t < 4 \\ 5+4t-t^2 & \text{if } 4 < t < 5 \\ 0 & \text{if } 5 < t \end{cases}$$



■

3.3 Properties of Convolution

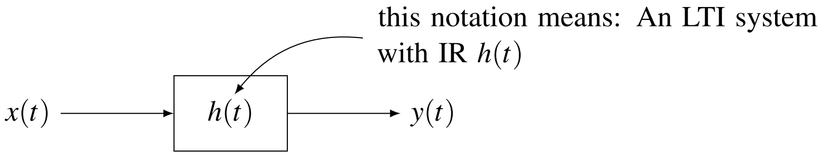
1. **Commutative**:

$$(x*h)(t) = (h*x)(t) \iff \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

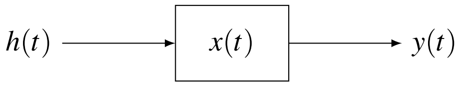
Indeed start with the LHS and use the change of variables $t-\tau = s$, $-d\tau = ds$:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{+\infty}^{-\infty} x(t-s)h(s)(-ds)\\ &= \int_{-\infty}^{\infty} h(s)x(t-s)ds\\ &= \text{RHS} \quad (s \text{ or } \tau \text{ are dummy variables})\end{aligned}$$

Implication:

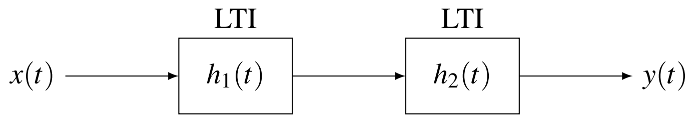


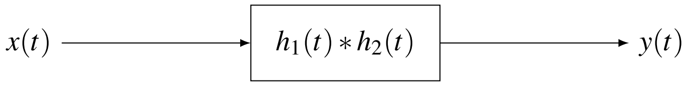
this notation means: An LTI system with IR $h(t)$



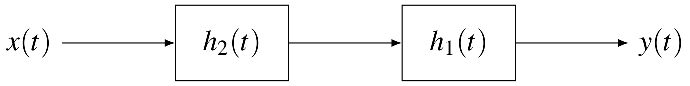
2. **Associative**:

$$[x(t)*h_1(t)]*h_2(t) = x(t)*[h_1(t)*h_2(t)]$$





Thus, the concatenation of 2 LTIs is an LTI with IR $h_1(t)*h_2(t)$. But since $*$ is also commutative, we also have:



i.e. we can interchange the order of two (or more) LTI systems!

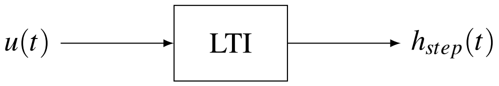
Proof.

$$\begin{aligned}\text{LHS} &= [x(t)*h_1(t)]*h_2(t)\\ &= z(t)*h_2(t)\\ \text{RHS} &= x(t)*[h_1(t)*h_2(t)]\\ &= x(t)*w(t)\end{aligned}$$

Write down convolution integrals and do change of variables...boring... □

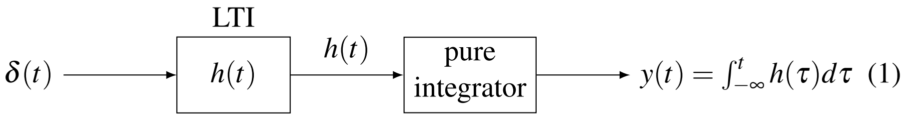
Implication of this property:

Definition 3.2 We define the step response (SR) of an LTI system as the output of the system when the input is the unit step function, i.e.

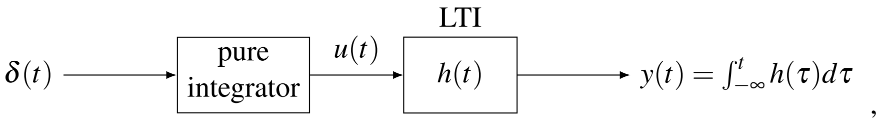


Q: How can we find the SR, i.e. $h_{step}(t)$ from the IR $h(t)$?

Consider the following system:



This is equivalent to



so $y(t) = h_{step}(t)$ (2). From (1) and (2), we get

$$h_{step}(t) = \int_{-\infty}^{t} h(\tau)d\tau \tag{3.3}$$

In the same way that $u(t)$ is the integral of $\delta(t)$, $h_{step}(t)$ is the integral of $h(t)$. This implies that

$$h(t) = \frac{d}{dt}h_{step}(t) \tag{3.4}$$

We can use the above to evaluate the IR of a system experimentally by first evaluating $h_{step}(t)$ (which is easier, since $u(t)$ may be easier to produce in the lab).

Careful: We need to show that the pure integrator is an LTI system!

$$\begin{aligned} x(t) &= \alpha_1 x_1(t) + \alpha_2 x_2(t) \\ y(t) &= \int_{-\infty}^{t} [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]\, d\tau = \alpha_1 \int_{-\infty}^{t} x_1(\tau) + \alpha_2 \int_{-\infty}^{t} x_2(\tau)d\tau \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Also, for $x(t-t_0)$ as input

$$\int_{-\infty}^{t} x(\tau - t_0)\, d\tau = \int_{-\infty}^{t-t_0} x(s)ds = y(t-t_0),$$

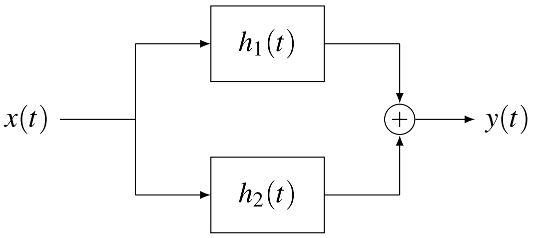
where the change of variables $s = \tau - t_0, ds = d\tau$ was used.
Alternatively: The pure integrator is described by the LCCDE

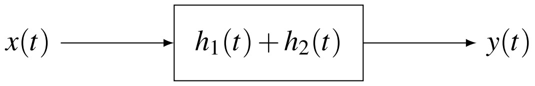
$$y'(t) = x(t),\ y(-\infty) = 0$$

so it is LTI.

3. **Distributive:**

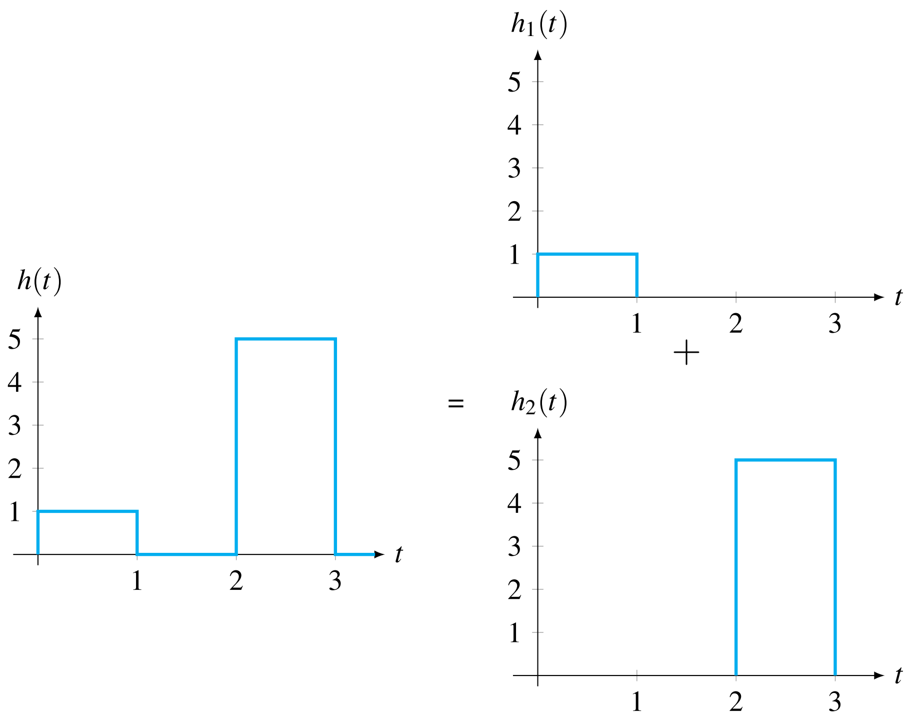
$$[x(t) * h_1(t)] + [x_2(t) * h_2(t)] = x(t) * [h_1(t) + h_2(t)]$$





Proof. Trivial due to the linearity of $\int$. Do it! □

■ Example 3.6



So $h(t) = h_1(t) + h_2(t) = h_1(t) + 5h_1(t-2)$.
For any signal $x(t)$, we have

$$\begin{aligned} x(t) * h(t) &= x(t) * [h_1(t) + h_2(t)] \\ &= x(t) * h_1(t) + x(t) * h_2(t) \\ &= y_1(t) + y_2(t) \end{aligned}$$

AND also

$$y_2(t) = 5y_1(t-2)$$

since

$$h_1(t) \longrightarrow \boxed{x(t)} \longrightarrow y_1(t)$$

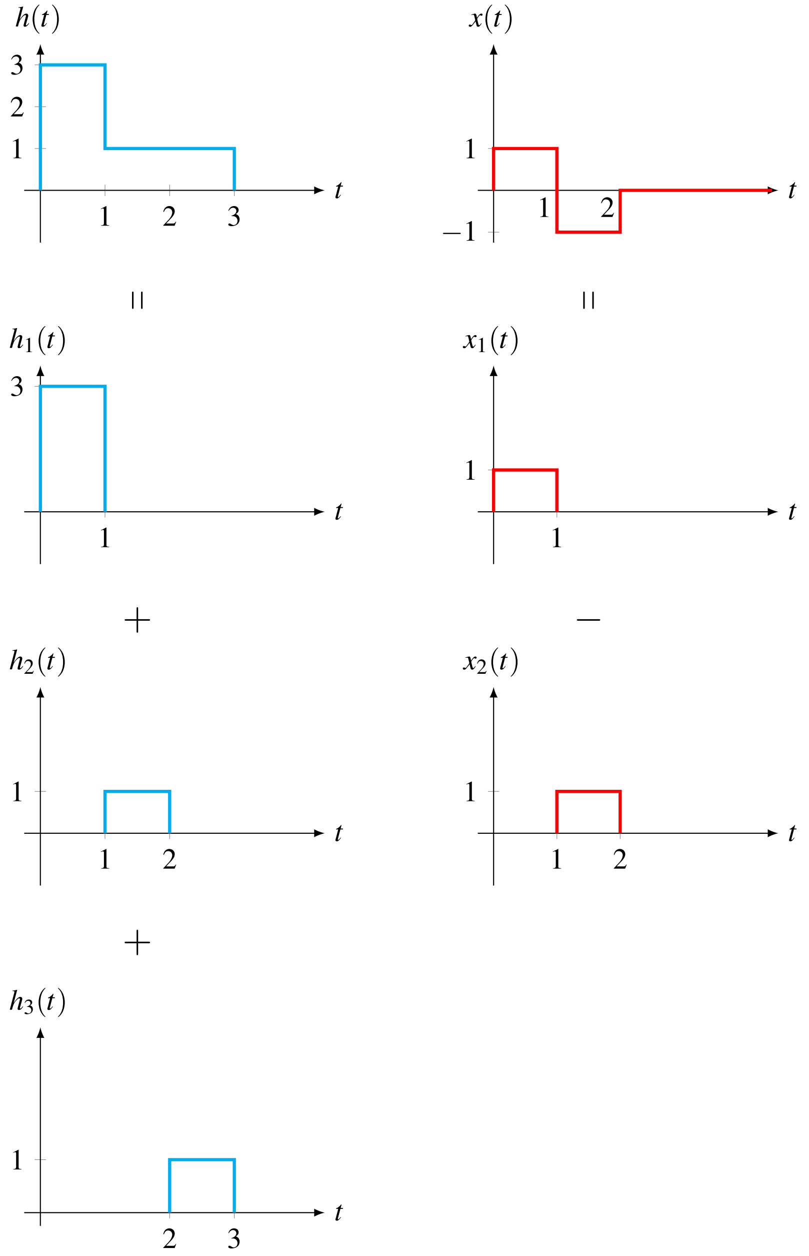
$$5h_1(t-2) \longrightarrow \boxed{x(t)} \longrightarrow 5y_1(t-2)$$

Overall:

$$x(t) * h(t) = y_1(t) + 5y_1(t-2)$$

so we only need to evaluate a simple convolution of $x(t)$ with a square pulse. ■

■ Example 3.7



$$\begin{aligned}
h(t) &= h_1(t) + \frac{1}{3}h_1(t-1) + \frac{1}{3}h_1(t-2) \\
x(t) &= x_1(t) - x_1(t-1) \\
h(t)*x(t) &= 6 \text{ terms (call } y_1(t) \triangleq h_1(t)*x_1(t)) \\
&= y_1(t) + \frac{1}{3}y_1(t-1) + \frac{1}{3}y_1\cancel{(t-2)} \\
&\quad - y_1(t-1) - \frac{1}{3}y_1\cancel{(t-2)} - \frac{1}{3}y_1(t-3) \\
&= y_1(t) - \frac{2}{3}y_1(t-1) - \frac{1}{3}y_1(t-3)
\end{aligned}$$

where $y_1(t) = x_1(t) * h_1(t)$. ■

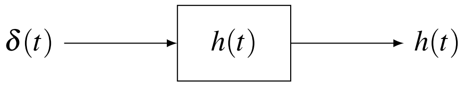
4. **Convolution with Delta's:**

$$\delta(t-t_0) * h(t) = h(t-t_0) \tag{3.5}$$

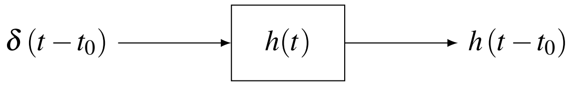
Specifically:

$$\delta(t) * h(t) = h(t)$$

Def. of $h(t)$:



(TI)



3.4 Causality and BIBO Stability of LTI Systems

We can check if an LTI system is causal/BIBO stable by just looking at its IR.

Proposition 3.2 An LTI system is causal if and only iff $h(t) = 0$, $\forall t < 0$ (for all $t < 0$).

Proof.

- Recall that a system is causal if $y(t)$ cannot depend on $x(\tau)$ for $\tau > t$.
- For an LTI system, we always have

$$\begin{aligned}
y(t) = x(t)*h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau + \int_{t}^{+\infty} x(\tau)h(t-\tau)d\tau
\end{aligned}$$

where the first term is the causal part and the second term is the non-causal part. Consider the non-causal part. By using the change of variables $t - \tau = s$, one can rewrite the non-causal part as:

$$\int_{t}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{0} x(t-s)h(s)ds$$

i) If $h(s) = 0$, $\forall s < 0$ (for all $s < 0$), then the non-causal part $= 0$ and

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

which means the system is causal.

ii) If the system is causal then for every $x(t)$ the non-causal part

$$\int_{-\infty}^{0} x(t-s)h(s)ds$$

must be zero. Since this has to be true ***for all*** $x(t)$, this implies that $h(s) = 0$, for all $s < 0$.

$\square$

By the way, for an LTI system to be memoryless we need that $y(t)$ is only a function of $x(t)$. Since $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, this implies that $h(t-\tau)$ has to be 0 for all $\tau < t$ and for all $\tau > t$; i.e. $h(s)$ has to be 0 for all $s > 0$ and for all $s < 0$. The only non-zero signal satisfying that is a **Dirac delta** (or **its derivatives**).

$$\begin{aligned} x(t) \to y(t) &= x(t) * \overset{k\delta(t)}{\cancel{h(t)}} \\ &= kx(t) \end{aligned}$$

R *Careful:*

$$x(t) \to y(t) = \frac{dx(t)}{dt}$$

is ***NOT memoryless*** since you cannot determine $y(t)$ from $x(t)$ alone:

$$y(t) = \lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t)}{\varepsilon}$$

Proposition 3.3 An LTI system is BIBO stable **if and only if** $h(t)$ is "***absolutely integrable***" i.e.

$$\int_{-\infty}^{\infty} |h(t)|dt = M < \infty \tag{3.6}$$

Proof.

i) Suppose $h(t)$ is absolutely integrable (1), and consider any bounded input $x(t)$, i.e. $|x(t)| < M_1$, for all t (2).

$$\begin{aligned} |y(t)| &\overset{\text{LTI}}{=} |x(t) * h(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)|d\tau \\ &\overset{(2)}{\leq} M_1 \int_{-\infty}^{\infty} |h(\tau)|d\tau \\ &\overset{(1)}{\leq} M_1 M \end{aligned}$$

so the output is also bounded $\implies$ the system is BIBO stable.

ii) Conversely, suppose (1) does not hold, i.e.

$$\int_{-\infty}^{\infty} |h(t)|dt = \infty \tag{3}$$

and consider the input

$$x(t) = \frac{h^*(-t)}{|h^*(-t)|} \qquad (4) \tag{4}$$

- $x(t)$ is bounded:

$$|x(t)| = \left|\frac{h^*(-t)}{|h^*(-t)|}\right| = \frac{|h^*(-t)|}{|h^*(-t)|} = 1 < \infty$$

- But:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

In particular:

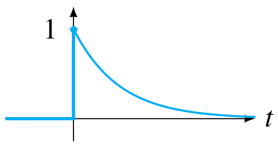
$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} x(-\tau)h(\tau)d\tau \overset{(4)}{=} \int_{-\infty}^{\infty} \frac{h^*(\tau)}{|h^*(\tau)|}h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{|h(\tau)|^{\cancel{2}}}{\cancel{|h(\tau)|}}d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau \overset{(3)}{=} \infty \end{aligned}$$

so $y(t)$ is NOT bounded, so the system is NOT BIBO stable.

□

■ **Example 3.8** Consider an LTI system with impulse response

$$h(t) = e^{-t}u(t)$$



The system is known to be causal. Is it BIBO stable?

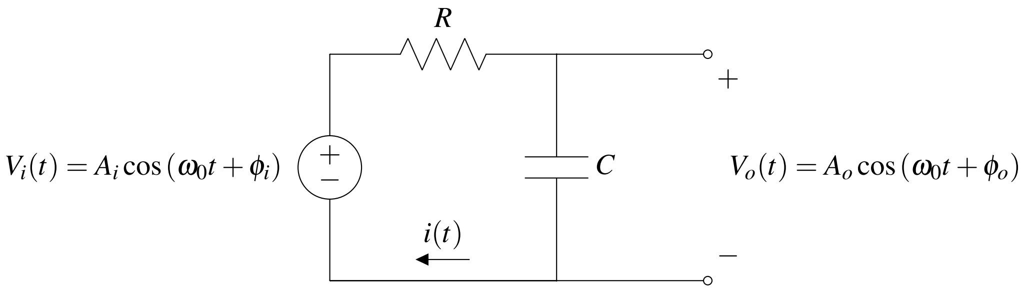
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} \left|e^{-t}u(t)\right| dt = \int_{0}^{\infty} e^{-t}dt = -e^{-t}\Big|_{0}^{+\infty} = 1 < \infty$$

so YES it is BIBO-stable. ■

3.5 LTI Response to Complex Exponentials and Transfer Function

- So few of you have seen how to evaluate $y(t)$ from $x(t) * h(t)$.
- Recall from EECS 215, RLC circuits, voltage/current when sources are sinusoidal signals

■ **Example 3.9**



When the input is a cosine, the output is a cosine with the SAME frequency but different amplitude/phase. Using "phasors":

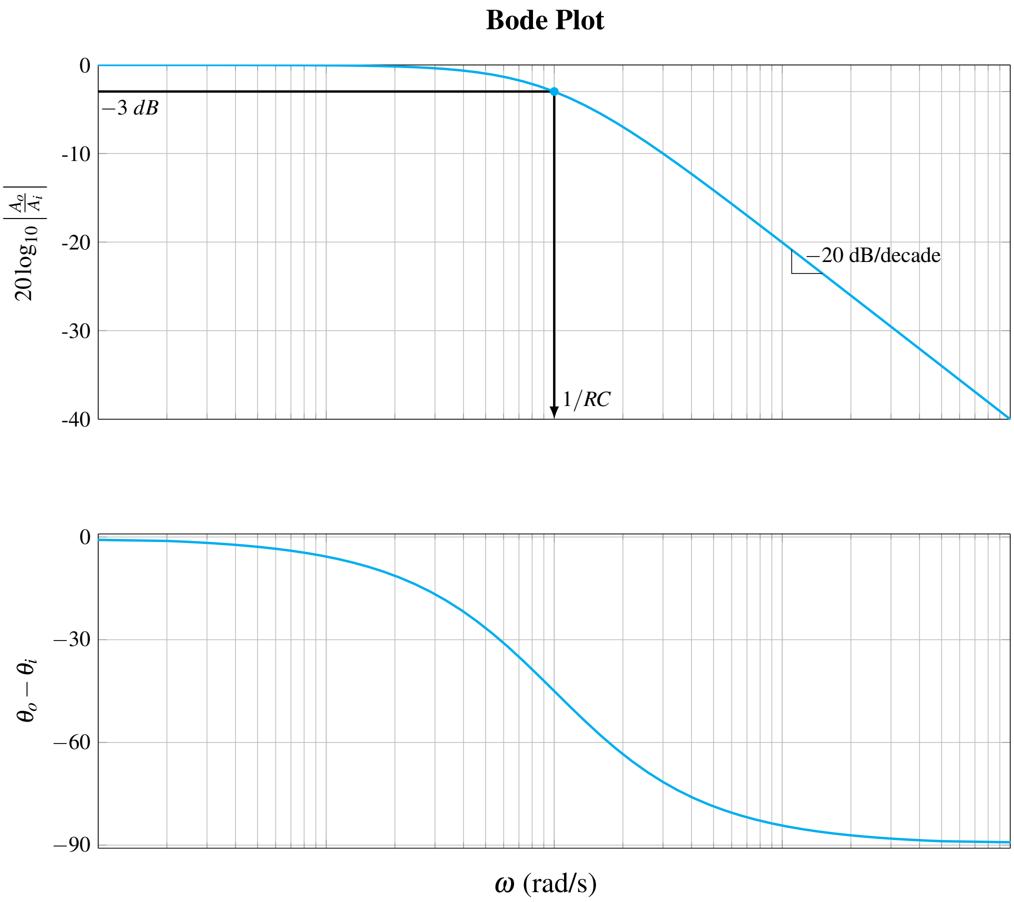
$$\text{Input} \leftrightarrow A_i e^{j\theta_i}$$

$$\text{Output} \leftrightarrow A_o e^{j\theta_o}$$

For this example,

$$A_o e^{j\theta_o} = A_i e^{j\theta_i} \frac{1/(j\omega_0 C)}{1/(j\omega_0 C) + R} = A_i e^{j\theta_i} \frac{1}{1 + j\omega_0 RC}$$

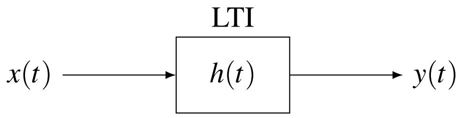
Bode Plots:



■

As it turns out, it is **NOT a COINCIDENCE**. The output of a sinusoidal signal with frequency ω_0 is itself a sinusoidal signal with the same frequency (but different amplitude/phase). This is a **GENERAL PROPERTY of LTI Systems**!

Consider an LTI system with IR $h(t)$



Consider $x(t) = e^{j\omega t}$, $\forall t$ (for all t), where ω is a *fixed frequency.*
$y(t) = ?$

Using convolution,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau$$
$$= \int_{-\infty} e^{j\omega t}e^{-j\omega\tau}h(\tau)d\tau$$

Since $e^{j\omega t}$ doesn't depend on τ, it can be taken out of the integral:

$$y(t) = e^{j\omega t}\int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau$$

Definition 3.3 For an LTI system with IR $h(t)$, we define the frequency response function (FRF) as:

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \tag{3.7}$$

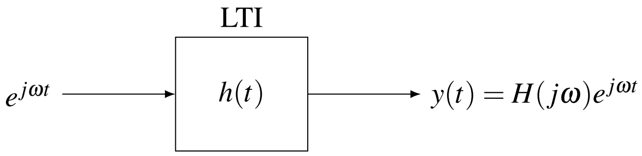
Why is it $H(j\omega)$ and not just $H(\omega)$? Weird...
Later we will define another function (transfer function)

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau,\ \forall s \in \mathbb{C} \text{ (for any } s \in \mathbb{C}) \tag{3.8}$$

where $s = \sigma + j\omega$, σ is the real part, and ω is the imaginary part. So for $\sigma = 0$ we get

$$\text{TF} = H(s)|_{s=0+j\omega} = H(j\omega) = \text{FRF}$$

What did we just show?

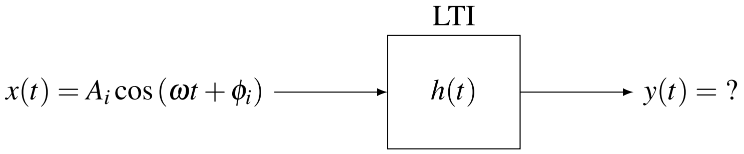


where $H(j\omega)$ is a complex number which is constant with respect to t.

$\boxed{\text{Complex Exponential In} \longrightarrow \text{Complex Exponential Out}}$

This is a remarkable fact! It is not true for other types of signals, and it is not true for non-LTI systems!

Now let's ask,



We will use the previous result together with trigonometry and linearity. We know:

$$\begin{aligned} A_i \cos(\omega t + \phi_i) &= A_i \frac{e^{j(\omega t + \phi_i)} + e^{-j(\omega t + \phi_i)}}{2} \\ &= \left(\frac{A_i}{2} e^{j\phi_i}\right) e^{j\omega t} + \left(\frac{A_i}{2} e^{-j\phi_i}\right) e^{j(-\omega)t} \end{aligned}$$

Due to linearity, the overall output is

$$\begin{aligned} y(t) &= \frac{A_i}{2} e^{j\phi_i} H(j\omega) e^{j\omega t} + \frac{A_i}{2} e^{-j\phi_i} H(-j\omega) e^{-j\omega t} \\ &= \text{how can we further simplify?} \end{aligned}$$

For this we need some properties of the FRF.

3.6 Properties of the Frequency Response Function

$$\boxed{\text{For now assume that } h(t) \text{ is real.}}$$

Property 1: In general $H(j\omega)$ is a complex number.

Proof.

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} h(\tau) \left[\cos(\omega\tau) - j \sin(\omega\tau)\right] d\tau \\ &= \left[\int_{-\infty}^{\infty} h(\tau) \cos(\omega\tau) d\tau\right] - j \left[\int_{-\infty}^{\infty} h(\tau) \sin(\omega\tau) d\tau\right] \end{aligned}$$

where the first term is real and the second term is imaginary. □

Property 2:

- $|H(j\omega)|$ is even
- $\angle H(j\omega)$ is odd

Proof.

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Take the complex conjugate of both sides:

$$\begin{aligned} H^*(j\omega) &= \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)^* \\ &= \int_{-\infty}^{\infty} h^*(\tau)e^{j\omega\tau}d\tau \end{aligned}$$

Since we are assuming $h(\tau)$ is real,

$$\begin{aligned} H^*(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j(-\omega)\tau}d\tau \\ &= H(-j\omega) \end{aligned}$$

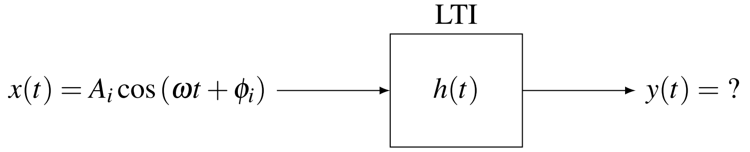
so

$$\begin{tikzcd}
 & H^*(j\omega) = H(-j\omega) \arrow[dl] \arrow[dr] & \\
|H^*(j\omega)| = |H(-j\omega)| \arrow[d, Rightarrow] & & \angle H^*(j\omega) = \angle H(-j\omega) \arrow[d, Rightarrow] \\
|H(j\omega)| = |H(-j\omega)| & & -\angle H(j\omega) = \angle H(-j\omega) \\
\text{(even)} & & \text{(odd)} \\
 & \text{this property is called "Hermitian" symmetry and is a generalization of even/odd symmetry to complex functions.} \arrow[uuuu] &
\end{tikzcd}$$

□

Back to our problem:

How to simplify $y(t)$



$$\begin{aligned} y(t) &= \frac{A_i}{2}e^{j\phi_i}H(j\omega)e^{j\omega t} + \frac{A_i}{2}e^{-j\phi_i}H(-j\omega)e^{-j\omega t} \\ &= \frac{A_i}{2}e^{j\phi_i}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} + \frac{A_i}{2}e^{-j\phi_i}|H(-j\omega)|e^{j\angle H(-j\omega)}e^{-j\omega t} \\ &= \frac{A_i}{2}e^{j\phi_i}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} + \frac{A_i}{2}e^{-j\phi_i}|H(j\omega)|e^{-j\angle H(j\omega)}e^{-j\omega t} \\ &= A_i|H(j\omega)|\frac{e^{j[\omega t+\phi_i+\angle H(j\omega)]} + e^{-j[\omega t+\phi_i+\angle H(j\omega)]}}{2} \\ &= A_i|H(j\omega)|\cos(\omega t + \phi_i + \angle H(j\omega)) \end{aligned} \tag{3.9}$$

3.7 Summary

For LTI systems: where $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$ is the FRF of the LTI system.[2]

3.8 Computing the FRF for an LTI Described by an LCCDE

■ **Example 3.10** Consider a BIBO-stable LTI system described by the following LCCDE:

$$y''(t) + 3y'(t) + 2y(t) = 4x(t) + 5x'(t)$$

Q: How can we evaluate $H(j\omega)$?
A: Find $h(t)$ from the LCCDE and then

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Hard! Instead, there is a much easier way! Since this is an LTI BIBO-stable, when the input is $x(t) = e^{j\omega t}$, the output is $y(t) = H(j\omega)e^{j\omega t}$:

$$\begin{aligned} x'(t) &= j\omega e^{j\omega t} \\ y'(t) &= H(j\omega)(j\omega)e^{j\omega t} \\ y''(t) &= H(j\omega)(j\omega)^2 e^{j\omega t} \end{aligned}$$

Substituting these into the LCCDE,

$$\begin{aligned} (j\omega)^2 H(j\omega)\cancel{e^{j\omega t}} + 3j\omega H(j\omega)\cancel{e^{j\omega t}} + 2H(j\omega)\cancel{e^{j\omega t}} &= 4\cancel{e^{j\omega t}} + 5j\omega\cancel{e^{j\omega t}} \\ \implies H(j\omega)\left[(j\omega)^2 + 3j\omega + 2\right] &= 4 + 5j\omega \\ \implies H(j\omega) &= \frac{4+5j\omega}{(j\omega)^2 + 3j\omega + 2} \end{aligned}$$

■

Following exactly the same procedure for the general case

$$\sum_{k=0}^{n} a_k \frac{d^{(k)}y(t)}{dt^k} = \sum_{l=0}^{m} b_l \frac{d^{(l)}x(t)}{dt^l}$$

we get

$$H(j\omega) = \frac{\sum_{l=0}^{m} b_l (j\omega)^l}{\sum_{k=0}^{n} a_k (j\omega)^k} \tag{3.10}$$

R **Eigenvalues/Eigenvectors:**

---

[2] All these results assumed that $H(j\omega)$ is well defined, i.e. the integral

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

exists. This is true if $h(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

Recall that this is also the condition for BIBO stability. So FRF exists if the system is BIBO stable.

Definition 3.4 For a square $n \times n$ matrix A, we say that the $n \times 1$ vector $\vec{x}$ is an eigenvector (e-vector) of A with corresponding eigenvalue (e-value) $\lambda \in \mathbb{C}$ if

$$A \cdot \vec{x} = \lambda \vec{x} \tag{3.11}$$

■ **Example 3.11**

$$A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$$

then

$$\vec{x}_1 = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}, \ \lambda_1 = 1$$

and

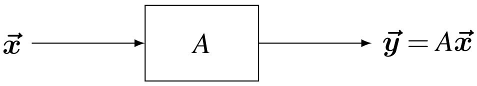
$$\vec{x}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \ \lambda_2 = 4$$

are two pairs of e-vectors/e-values of A. Indeed,

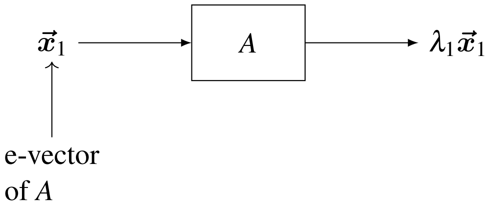
$$A\vec{x}_1 = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} - \sqrt{2} \\ 2 - 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} = \lambda_1 x_1$$

■

You can think of $\vec{y} = A\vec{x}$ as a system

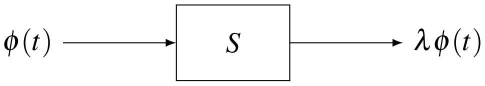


Then the e-vector has the property that if it is the input to this system, the output will be a scaled version of the input (with scaling factor the corresponding e-value)

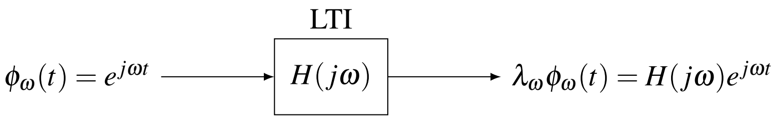


For continuous-time signals and systems, eigenfunctions (e-functions) are analogous to e-vectors.

Definition 3.5 For a system S with input $x(t)$ and output $y(t)$, we say that $\phi(t)$ is an e-function of S (with corresponding e-value λ) if



What we showed earlier is that for LTI (BIBO-stable) systems, complex exponentials $\phi_\omega(t) = e^{j\omega t}$ are e-functions with corresponding e-values $\lambda_\omega = H(j\omega)$.



Q: Why are e-functions of S important?
A: *Suppose* $\phi_1(t)$, $\phi_2(t)$, $\ldots$, $\phi_k(t)$, $\ldots$ are e-functions of S with corresponding e-values λ_1, λ_2, $\ldots$, λ_k, $\ldots$

$$S[\phi_k](t) = \lambda_k \phi_k(t) \tag{3.12}$$

- *Suppose* the input $x(t)$ can be written as a linear combination of e-functions, e.g.

$$x(t) = \sum_k a_k \phi_k(t) \tag{3.13}$$

- *Suppose* the system S is linear.
- Then the output $y(t)$ is

$$\begin{aligned} y(t) = S[x](t) &= S\left[\sum_k a_k \phi_k\right](t) && (3.14) \\ \text{(linearity)} &= \sum_k a_k S[\phi_k](t) && (3.15) \\ \text{(e-function)} &= \sum_k a_k \lambda_k \phi_k(t) && (3.16) \end{aligned}$$

  which is also a linear combination of ϕ_k's with coefficients $b_k = \lambda_k a_k$.
- No need for complicated calculations to find $y(t)\ldots$ (as long as we know e-functions, e-values, and coefficients a_k's).

4. Fourier Series

Motivation: Consider an LTI system with FRF $H(j\omega)$.

$$e^{j\omega t} \longrightarrow \boxed{S} \longrightarrow H(j\omega)e^{j\omega t}$$

or

$$\cos(\omega t+\phi) \longrightarrow \boxed{S} \longrightarrow |H(j\omega)|\cos(\omega t+\phi+\angle H(j\omega))$$

and using linearity:

$$A_1e^{j\omega_1 t}+A_2e^{j\omega_2 t} \longrightarrow \boxed{S} \longrightarrow A_1H(j\omega_1)e^{j\omega_1 t}+A_2H(j\omega_2)e^{j\omega_2 t}$$

or

$$\begin{array}{l} A_1\cos(\omega_1 t+\phi_1) \\ \quad +A_2\cos(\omega_2 t+\phi_2) \end{array} \longrightarrow \boxed{S} \longrightarrow \begin{array}{l} A_1|H(j\omega_1)|\cos(\omega_1 t+\phi_1+\angle H(j\omega_1)) \\ \quad +A_2|H(j\omega_2)|\cos(\omega_2 t+\phi_2+\angle H(j\omega_2)) \end{array}$$

Q: Wouldn't it be nice if we could express (approximately) any signal $x(t)$ as a linear combination of complex exponentials?
e.g.

$$x(t)\approx\sum_k A_k e^{j\omega_k t}$$

Then using linearity,

$$x(t)\approx\sum_k A_k e^{j\omega_k t} \longrightarrow \boxed{S} \longrightarrow y(t)=\sum_k A_k H(j\omega_k)e^{j\omega_k t}$$

so evaluation of $y(t)$ is a simple task (no need for complicated convolution!)

A: Guess what: This can be done *exactly* (i.e. no approximation) for *any* periodic signal $x(t)$.

$\Longrightarrow$ Fourier Series

4.1 Fourier Series *(Complex Exponential Form)*

Joseph Fourier 1768 – 1830

Proposition 4.1 — Fourier Series. Consider a piecewise continuous periodic function $x(t)$ (real or complex) with period T_0. Then $x(t)$ can be represented **exactly** as a linear combination of infinitely many complex exponentials:

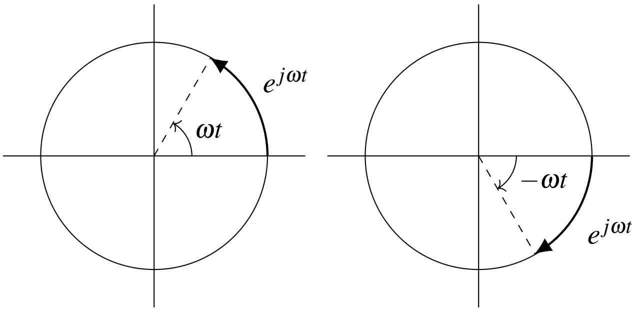
$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_0 nt}, \quad \omega_0 = \frac{2\pi}{T_0}} \tag{4.1}$$

for appropriately defined constants $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$.

The representation in Equation (4.1) is called the "complex exponential Fourier Series" and the coefficients $\{c_n\}_{n=-\infty}^{+\infty}$ are called the "Complex Fourier Coefficients."

Several Remarks:

R The series contains complex exponentials with positive and negative frequencies. Recall



Both positive and negative frequency complex exponentials are required to represent real signals. E.g.

$$\cos(\omega_t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

R

- The angular frequency $\omega_0 = \frac{2\pi}{T_0}$ is called the ***fundamental frequency***.
- The $\pm n^{\text{th}}$ term in the expansion $c_n e^{j\omega_0 nt}$ and $c_{-n} e^{-j\omega_0 nt}$ involves a complex exponential with frequency $n\omega_0$ and $-n\omega_0$. $\pm n\omega_0$ is called the n^{th} harmonic.
- The term corresponding to $n = 0$ is $c_0 e^{j\omega_0 \cdot 0t} = c_0$ is called the **DC term**.

R There are some mild conditions for $x(t)$ to have a Fourier Series representation:

- $x(t)$ has to be piecewise continuous
- At the points of discontinuity of $x(t)$ we can write

$$\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} = \underset{\substack{\uparrow \\ \text{average value of the} \\ \text{right and left limits} \\ \text{at the discontinuity}}}{\frac{x(t^+) + x(t^-)}{2}}$$

So we learned that every piecewise continuous periodic function can be represented by an infinite sum of complex exponentials with frequencies being integer multiples of the fundamental frequency.

Q: But what are the appropriate coefficients c_n?

A: Let's evaluate the m^{th} coefficient c_m

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{n=\infty} c_n e^{j\omega_0 nt} \\
\implies \int_0^{T_0} x(t)e^{-jm\omega_0 t}dt &= \int_0^{T_0} \left(\sum_{n=-\infty}^{n=\infty} c_n e^{j\omega_0 nt} \right) e^{-jm\omega_0 t}dt \\
&= \sum_{n=-\infty}^{\infty} c_n \int_0^{T_0} e^{j(n-m)\omega_0 t}dt \qquad (4.2)
\end{aligned}
$$

where we need to evaluate the integral in Equation (4.2).

- *Case 1:* If $n = m$

$$\int_0^{T_0} 1dt = T_0$$

- *Case 2:* If $n \neq m$

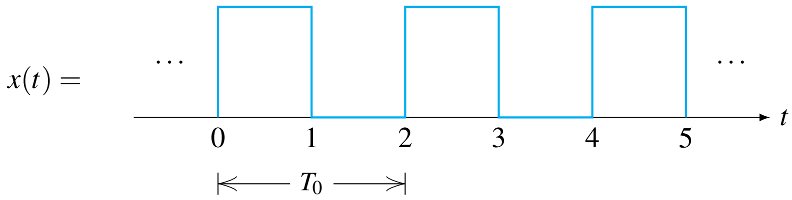
$$
\begin{aligned}
\int_0^{T_0} e^{j\omega_0(n-m)t}dt &= \left. \frac{e^{j\omega_0(n-m)t}}{j\omega_0(n-m)} \right|_0^{T_0} \\
&= \frac{e^{j\omega_0(n-m)T_0} - e^{j\omega_0(n-m)\cdot 0}}{j\omega_0(n-m)} \\
&\overset{\omega_0=\frac{2\pi}{T_0}}{=} \frac{\cancelto{1}{e^{j2\pi(n-m)}} - 1}{j\omega_0(n-m)} \\
&= 0\,!
\end{aligned}
$$

Substitute back these numbers in Equation (4.2)

$$
\begin{aligned}
\int_0^{T_0} x(t)e^{-j\omega_0 mt}dt &= \sum_{n=-\infty}^{\infty} c_n \begin{cases} T_0 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \\
&= c_m T_0 \\
\implies & \boxed{c_m = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 mt}dt} \text{ for all } m = \cdots -2, -1, 0, 1, 2, \dots
\end{aligned}
$$

R You can perform this integral over any interval of duration T_0 since $x(t)e^{-j\omega_0 mt}$ is periodic with period T_0.

■ Example 4.1



$x(t)$ is periodic with period $T_0 = 2$, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$.

Then

$$\begin{aligned} c_n &= \frac{1}{T_0} x(t) e^{-j\omega_0 nt} dt \\ &= \frac{1}{2} \int_0^1 e^{-j\pi nt} dt \end{aligned}$$

- *Case 1:* If $n = 0$,

$$c_0 = \frac{1}{2} \int_0^1 1 dt = \frac{1}{2}$$

- *Case 2:* If $n \neq 0$,

$$c_n = \frac{1}{2} \frac{e^{-jn\pi t}}{-jn\pi} \bigg|_0^1 = \frac{e^{-jn\pi} - 1}{-j2n\pi}$$

  - *Case 2a:* If $n =$ even:

$$c_n = 0$$

  - *Case 2b:* If $n =$ odd:

$$c_n = \frac{-1-1}{-j2n\pi} = \frac{1}{jn\pi}$$

So overall

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{jn\pi} & \text{if } n = \text{odd} \end{cases}$$

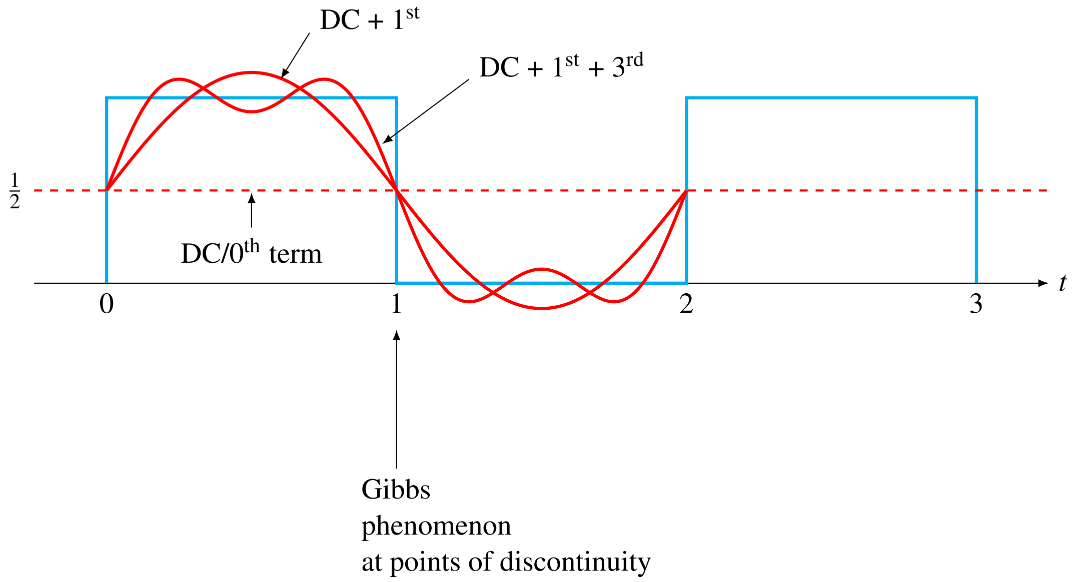
and

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi t} \\ &= c_0 + \underbrace{\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{-1} \frac{1}{jn\pi} e^{jn\pi t}}_{k=-n} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \frac{1}{jn\pi} e^{jn\pi t} \\ &= c_0 + \sum_{\substack{k=1 \\ k \text{ odd}}}^{+\infty} \frac{-1}{jk\pi} e^{-jk\pi t} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \frac{1}{jn\pi} e^{jn\pi t} \\ &= \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \frac{1}{\cancel{j}n\pi} \underbrace{\left(\frac{e^{jn\pi t} - e^{-jn\pi t}}{2j} \right)}_{\sin(n\pi t)} 2\cancel{j} \\ &= \frac{1}{2} + \sum\nolimits_{n \text{ odd}}^{n=1+\infty} \frac{2}{n\pi} sin(n\pi t) \\ &= \frac{1}{2} + \frac{2}{\pi} sin(\pi t) + \frac{2}{3\pi} \sin(3\pi t) + \frac{2}{5\pi} \sin(5\pi t) + \ldots \end{aligned}$$

■

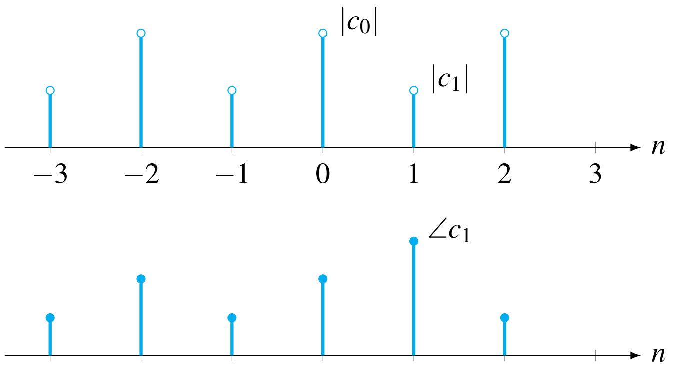
4.1.1 Demo

Consider a better and better approximation of $x(t)$ by including more and more terms of this series.



4.2 Magnitude/Phase Spectrum of Fourier Coefficients

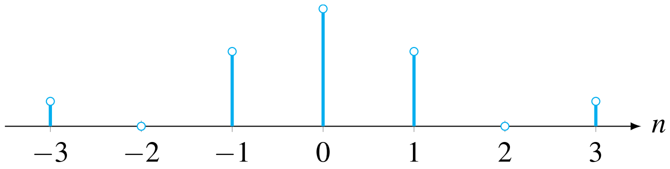
It is useful sometimes to plot the magnitude and phase of the Fourier coefficients, c_n.

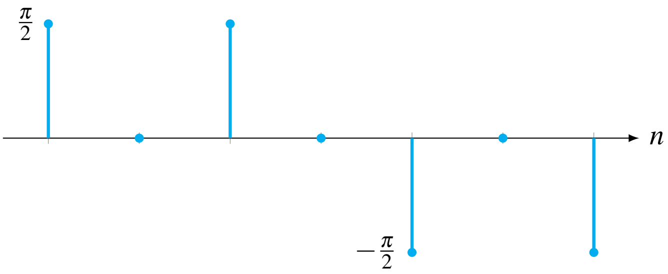


■ **Example 4.2** Previous Example

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{j\pi n} & \text{if } n = \text{odd} \end{cases} \implies |c_n| = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{\pi|n|} & \text{if } n = \text{odd} \end{cases}$$

$$\angle c_n = \begin{cases} 0 \;\; \text{if } n = \text{even} \\ \_\_ \;\; \text{if } n = \text{odd} \\ \angle \frac{1}{j\pi n} = \angle - \frac{j}{\pi n} = \begin{cases} \frac{\pi}{2} & \text{if } n < 0 \\ -\frac{\pi}{2} & \text{if } n > 0 \end{cases} \text{odd} \end{cases}$$





■

Observe in this example that $|c_n|$ is an even function of n and $\angle c_n$ is an odd function of n. This is not a coincidence:

Proposition 4.2 For a real signal $x(t)$, the Fourier coefficients satisfy the following:

$$\begin{tikzcd} & c_{-n} = c_n^* \qquad \forall n \text{ (for all } n\text{)} \arrow[dl] \arrow[dr] & \\ |c_{-n}| = |c_n^*| = |c_n| & & \angle c_{-n} = \angle c_n^* = -\angle c_n \end{tikzcd}$$

Proof. From definitions,

$$\begin{aligned} c_n &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 nt}dt \\ \Longrightarrow \quad c_n^* &= \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 nt}dt\right)^* = \frac{1}{T_0}\int_0^{T_0} \underset{\text{real}}{x^*(t)}e^{j\omega_0 nt}dt \\ &\overset{\text{def}}{=} c_{-n} \end{aligned}$$

□

4.3 Trigonometric Form of Fourier Series

When $x(t)$ is real, it is more useful sometimes to represent $x(t)$ by a weighted sum of harmonically related sin and cos functions.

$$x(t) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(\omega_0 nt) + \sum_{n=1}^{\infty} b_n \sin(\omega_0 nt) \tag{4.3}$$

where the coefficients a_0, a_1, …, b_1, b_2, … are real-valued. This is called the ***Trigonometric Fourier Series***.

How is this related to the complex-exponential Fourier Series?

Recall the complex-exponential Fourier Series

$$\begin{aligned}
\underset{\text{real}}{x(t)} &= \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \\
&= c_0 e^{j0\omega_0 t} + \underbrace{\sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}}_{m=-n} + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} \\
&= c_0 + \sum_{m=1}^{+\infty} c_{-m} e^{-jm\omega_0 t} + \sum_{n=1}^{+\infty} c_n e^{jn\omega_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\omega_0 t}\right)^* + \left(c_n e^{jn\omega_0 t}\right)
\end{aligned}$$

Using the identity $\alpha + \alpha^* = 2\,\mathrm{Re}\{\alpha\}$, we get

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\left\{c_n e^{jn\omega_0 t}\right\} \tag{4.4}$$

Now using $c_n = \mathrm{Re}\{c_n\} + j\,\mathrm{Im}\{c_n\}$ and Euler's identity $e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(\omega_0 t)$, we can rewrite $x(t)$ as:

$$\begin{aligned}
x(t) &= c_0 + \sum_{n=1}^{+\infty} 2\left\{\mathrm{Re}[c_n]\cos(n\omega_0 t) - 2\,\mathrm{Im}[c_n]\sin(n\omega_0 t)\right\} \\
&= \underbrace{c_0}_{a_0} + \sum_{n=1}^{+\infty} \underbrace{2\,\mathrm{Re}\{c_n\}}_{a_n}\cos(n\omega_0 t) + \sum_{n=1}^{+\infty} \underbrace{-2\,\mathrm{Im}\{c_n\}}_{b_n}\sin(n\omega_0 t) \\
&= a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{+\infty} b_n \sin(n\omega_0 t)
\end{aligned} \tag{4.5}$$

The trigonometric Fourier Series representation contains cos and sin of harmonic frequencies $n\omega_0$, $n = 1, 2, 3, \ldots$. In the trigonometric Fourier Series,

$$\begin{aligned}
a_0 &= c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \in \mathbb{R} \\
a_n &= 2\,\mathrm{Re}\{c_n\} = 2\,\mathrm{Re}\left\{\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 nt}dt\right\} \\
&= \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega_0 t)\,dt \\
b_n &= -2\,\mathrm{Im}\{c_n\} = -2\,\mathrm{Im}\left\{\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 nt}dt\right\} \\
&= \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega_0 t)\,dt
\end{aligned}$$

■ **Example 4.3** In the previous example,

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{j\pi n} & \text{if } n = \text{odd} \end{cases}$$

so

$$a_0 = c_0 = \frac{1}{2}$$

$$a_n = 2\,\mathrm{Re}\{c_n\} = \begin{cases} 0 & \text{if } n = \text{even} \\ 0 & \text{if } n = \text{odd} \end{cases}$$

$$b_n = -2\,\mathrm{Im}\{c_n\} = \begin{cases} 0 \;\; n = \text{even} \\ -2\,\mathrm{Im}\left\{-\frac{j}{\pi n}\right\} = -2\frac{-1}{\pi n} = \frac{2}{\pi n} \end{cases}$$

so

$$x(t) = \frac{1}{2} + 0 + \sum_{\substack{n\,=\,1 \\ n\,=\,\text{odd}}}^{+\infty} \frac{2}{\pi n} \sin(n\overset{\frac{2\pi}{T_0} = \pi}{\nearrow\!\omega_0} t)$$

■

There is also a third way to express the Fourier Series of a real signal $x(t)$.

Consider the previous derivation, Equation (4.4),

$$\begin{aligned} x(t) = \cdots &= c_0 + \sum_{n=1}^{+\infty} 2\,\mathrm{Re}\{\underset{\overset{\uparrow}{|c_n|\,e^{j\angle c_n}}}{c_n} e^{jn\omega_0 t}\} \\ &= c_0 + \sum_{n=1}^{+\infty} 2\,\mathrm{Re}\left\{|c_n|\,e^{j\angle c_n} e^{jn\omega_0 t}\right\} \\ &= c_0 + \sum_{n=1}^{+\infty} 2\,|c_n|\cos\left(n\omega_0 t + \angle c_n\right) \end{aligned}$$

(polar-form Fourier Series) contains only cosines of harmonic frequencies ω_0, $2\omega_0$, $3\omega_0$, ….

■ Example 4.4

$$x(t) = \frac{1}{2} + \sum_{\substack{n\,=\,1 \\ n\,=\,\text{odd}}}^{+\infty} \frac{2}{\pi n} \cos\left(n\omega_0 t - \frac{\pi}{2}\right)$$

since

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{j\pi n} & \text{if } n = \text{odd} \end{cases}$$

■

4.4 Periodic Signals through LTI Systems

Consider a periodic signal $x(t)$ expressed through the Fourier Series as

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \\ &\overset{\text{(real)}}{=} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\omega_0 t\right) \\ &\overset{\text{(real)}}{=} c_0 + \sum_{n=1}^{\infty} 2\,|c_n|\cos\left(n\omega_0 t + \angle c_n\right) \end{aligned}$$

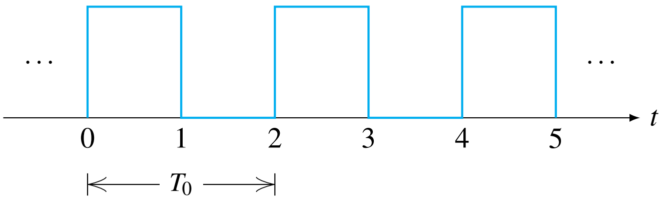
and a BIBO-stable LTI system with FRF $H(j\omega)$. The output $y(t)$ is given by

$$\begin{aligned}
y(t) &= \sum_{n=-\infty}^{+\infty} c_n H\left(jn\omega_0\right) e^{jn\omega_0 t} \\
&= c_0 H(j0) + \sum_{n=1}^{\infty} 2\left|c_n\right| \left|H\left(jn\omega_0\right)\right| \cos\left(n\omega_0 t + \angle c_n + \angle H\left(jn\omega_0\right)\right)
\end{aligned}$$

Note that each term in the series is affected differently because $H\left(jn\omega_0\right)$ depends on the harmonic frequency.

■ **Example 4.5** Periodic Square Wave through RC Circuit:

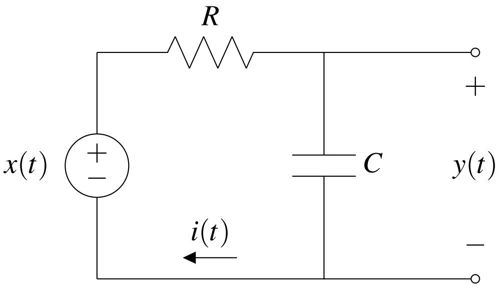
$x(t) =$



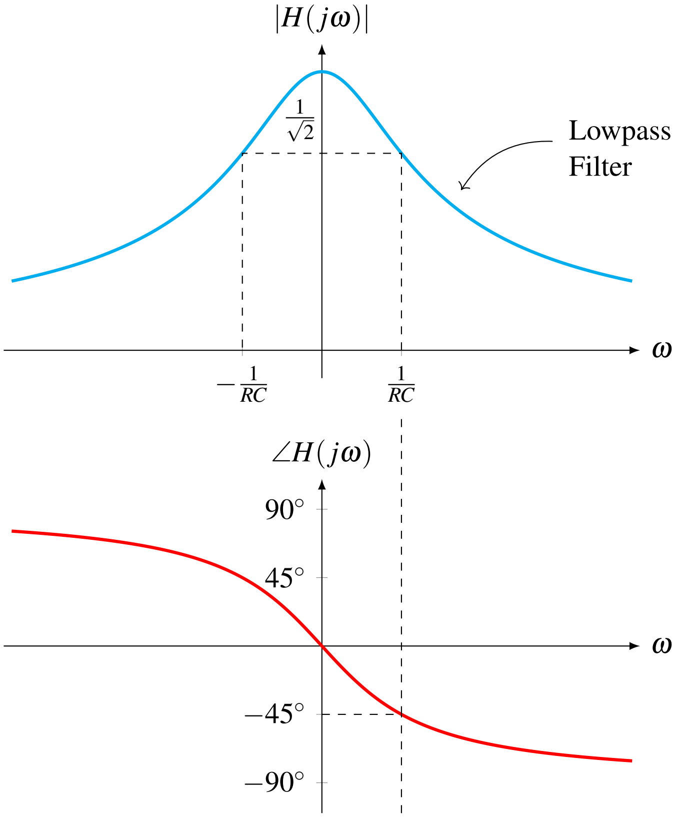
We saw earlier that

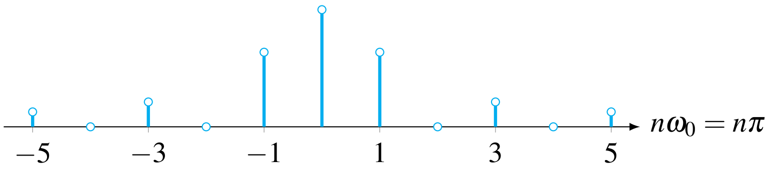
$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\cancelto{\pi}{\omega_0 t}},\ c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{j\pi n} & \text{if } n = \text{odd} \end{cases} \\
&= \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{2}{\pi n} \underbrace{\cos\left(n\omega_0 t - \frac{\pi}{2}\right)}_{\sin(n\omega_0 t)}
\end{aligned}$$

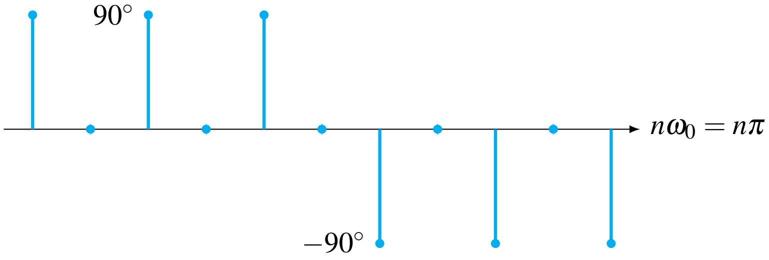
Consider the RC circuit



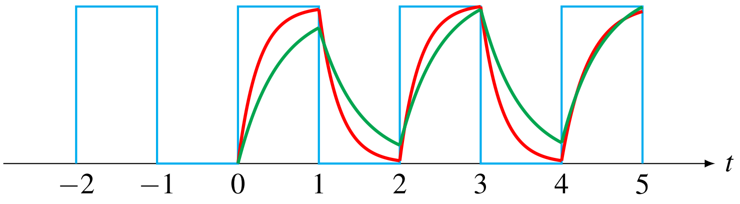
$$\begin{aligned}
H(j\omega) &= \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1 + RCj\omega} \\
|H(j\omega)| &= \frac{1}{\sqrt{1 + (RC\omega)^2}} \\
\angle H(j\omega) &= -\tan^{-1}(RC\omega)
\end{aligned}$$







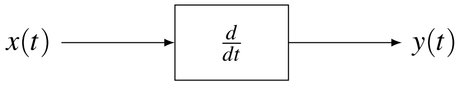
Output:



where the red and green curves represent the output for two different values of RC. See MATLAB Demo / (or real time GNURADIO). ■

■ **Example 4.6 Periodic signal through differentiator.**

Consider the system



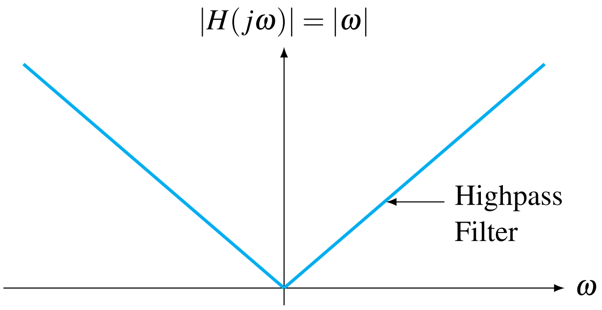
;

Q: Is it LTI?
A: YES, $y(t) = \frac{dx(t)}{dt}$ (LCCDE)

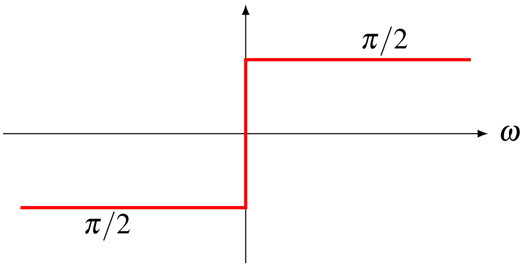
Q: What is the FRF?
A: Hard to evaluate $h(t)$ and then $H(j\omega)$ because $h(t) = \delta'(t)$!
Easier way:

$$
\begin{aligned}
x(t) &= e^{j\omega t} \\
y(t) &= x'(t) = \underbrace{(j\omega)}_{H(j\omega)} e^{j\omega t}
\end{aligned}
$$

so $H(j\omega) = j\omega$.



$$
\angle H(j\omega) = \begin{cases} \pi/2 & \omega > 0 \\ -\pi/2 & \omega < 0 \end{cases}
$$



If $x(t)$ is periodic then

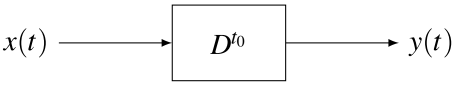
$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \\
y(t) &= \sum_{n=-\infty}^{\infty} c_n H\left(\right) \\
&= \sum_{n=-\infty}^{\infty} c_n H\left(jn\omega_0\right) e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} c_n \left(jn\omega_0\right) e^{jn\omega_0 t}
\end{aligned}$$

This can be verified by differentiating directly $x(t)$:

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} c_n \frac{d}{dt} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n \left(jn\omega_0\right) e^{jn\omega_0 t} \ \checkmark$$

∎

■ **Example 4.7 Periodic signal through a pure delay.**
Consider the system



;

Q: Is it LTI?
A: YES (proved it a long time ago)

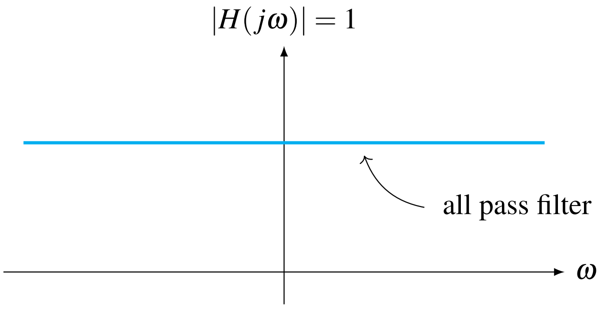
Q: What is its FRF?

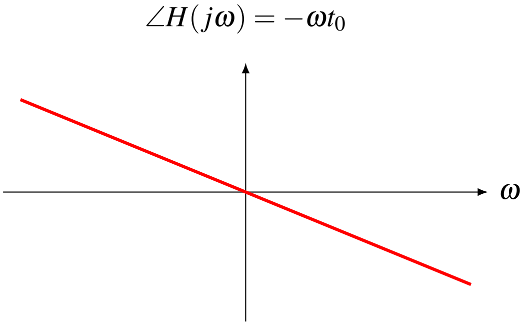
- *Method A:*

$$h(t) = \delta\left(t - t_0\right) \implies H(j\omega) = \int_{-\infty}^{\infty} \delta\left(t - t_0\right) e^{-j\omega t} dt = e^{-j\omega t_0}$$

- *Method B:*

$$x(t) = e^{j\omega t} \implies y(t) = e^{j\omega\left(t - t_0\right)} = \underbrace{e^{-j\omega t_0}}_{H(j\omega)} e^{j\omega t}$$





If $x(t)$ is periodic,

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} \\
y(t) &= \sum_{n=-\infty}^{\infty} c_n \underbrace{H\left(jn\omega_0\right)}_{e^{-jn\omega_0 t_0}} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0(t-t_0)} \\
&= x\left(t-t_0\right) \quad \text{as expected!}
\end{aligned}
$$

■

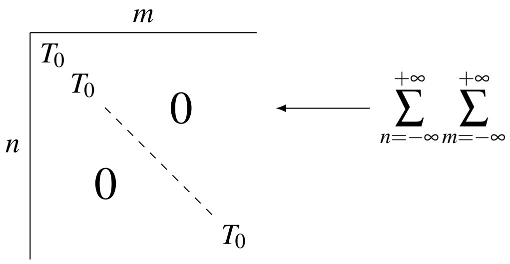
4.5 Parseval’s Theorem

Consider a periodic signal $x(t)$ with period T_0 and the Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

Q: What is the average power of $x(t)$?
A: We know that for a periodic signal,

$$\begin{aligned}
P_{avg} &= \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt \\
&= \frac{1}{T_0}\int_0^{T_0} \left(\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{+\infty} c_m e^{jm\omega_0 t}\right)^* dt \\
&= \frac{1}{T_0}\int_0^{T_0} \left(\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{+\infty} c_m^* e^{-jm\omega_0 t}\right) dt \\
&= \frac{1}{T_0}\int_0^{T_0} \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} c_n c_m^* e^{j(n-m)\omega_0 t}\,dt \\
&= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* \underbrace{\int_0^{T_0} e^{j(n-m)\omega_0 t}\,dt}_{\substack{\text{recall that we have}\\ \text{evaluated this!}\\ = \begin{cases} 0 & n \neq m \\ T_0 & n = m \end{cases}}}
\end{aligned}$$



so

$$P_{avg} = \frac{1}{\cancel{T_0}}\sum_{n=-\infty}^{+\infty} c_n c_n^* \cdot \cancel{T_0}$$

$\Longrightarrow$

Proposition 4.3 — Parseval's Theorem for Fourier Series.

$$\boxed{P_{avg} = \sum_{n=-\infty}^{+\infty} |c_n|^2} \tag{4.6}$$

Interpretation: each harmonic (i.e. the n^{th}) contributes $|c_n|^2$ to the total power. If $x(t)$ is real then $c_n = c_{-n}^* \implies |c_n|^2 = |c_{-n}|^2$, both the n^{th} and $(-n)^{\text{th}}$ harmonics contribute the same, so

$$P_{avg} = \underset{\substack{\uparrow\\ \text{DC power}}}{|c_0|^2} + \sum_{n=1}^{+\infty} \underbrace{2|c_n|^2}_{\substack{\text{power in the}\\ n^{\text{th}}\text{ harmonic}\\ \text{(positive +}\\ \text{negative)}}} \tag{4.7}$$

Compare this with the polar form of the Fourier Series

$$x(t) = \underset{\substack{\downarrow \\ \text{DC power} \\ |c_0|^2}}{c_0} + \sum_{n=1}^{+\infty} \underbrace{2\,|c_n|\cos\left(n\omega_0 t + \angle c_n\right)}_{\substack{A\cos(\omega t) \to \text{power } \frac{A^2}{2} \\ \text{so } \frac{(2|c_n|)^2}{2} = 2\,|c_n|^2}}$$

■ **Example 4.8** When approximating a signal $x(t)$ by a finite number of terms (say N),

$$x(t) \approx \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{N} 2\,|c_n|\cos\left(n\omega_0 t + \angle c_n\right)$$

we can choose N so that 99% of the power of the original signal is included in the approximation!

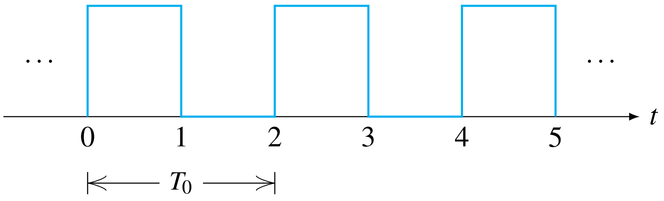
Choose N such that

$$|c_0|^2 + \sum_{n=1}^{N} 2\,|c_n|^2 \geq 0.99 \Bigg(\underset{\substack{\uparrow \\ \text{total power}}}{|c_0|^2 + \sum_{n=1}^{\infty} 2\,|c_n|^2} \Bigg)$$

A plot of $|c_n|^2$ *vs.* n or $n\omega_0$ is called a ***power spectrum***. ■

■ **Example 4.9** Square Wave:

$x(t) =$



$$P_{avg} = \frac{1}{2}\int_0^2 |x(t)|^2\,dt = \frac{1}{2}\int_0^1 1dt = \frac{1}{2} \tag{4.8}$$

Recall:

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n = \text{even} \\ \frac{1}{j\pi n} & \text{if } n = \text{odd} \end{cases}$$

From Parseval's Theorem,

$$\begin{aligned} P_{avg} &= |c_0|^2 + \sum_{n=1}^{+\infty} 2\,|c_n|^2 \\ &= \frac{1}{4} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{+\infty} 2\left(\frac{1}{\pi n}\right)^2 \\ &= \frac{1}{4} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{+\infty} \frac{2}{\pi^2 n^2} \end{aligned} \tag{4.9}$$

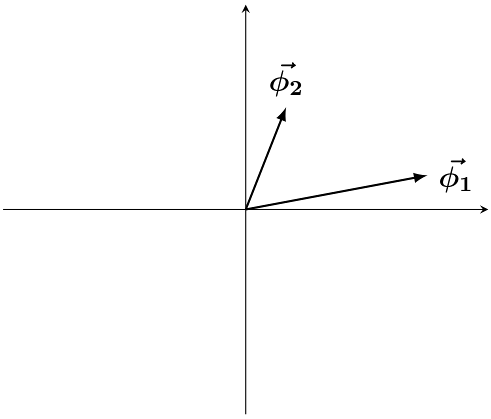
Since (4.8) = (4.9) $\implies$ we just discovered a nice algebraic expression or series:

$$\frac{1}{4} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{+\infty} \frac{2}{\pi^2 n^2} = \frac{1}{2} \iff \boxed{\sum_{\substack{n=1 \\ n=\text{odd}}}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{8}} \tag{4.10}$$

■

4.6 Geometric Interpretation of Fourier Series

Consider the 2D space $\mathbb{R}^2$ (our results generalize) and a system of coordinates defined through the vectors $\vec{\phi}_1$, $\vec{\phi}_2$.



Any vector $\vec{x} \in \mathbb{R}^2$ can be written as a weighted sum $\vec{x} = c_1\vec{\phi}_1 + c_2\vec{\phi}_2$.
e.g. $\vec{\phi}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{\phi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} = 2\vec{\phi}_1 + 3\vec{\phi}_2$$

How can we find these coefficients c_1, c_2?

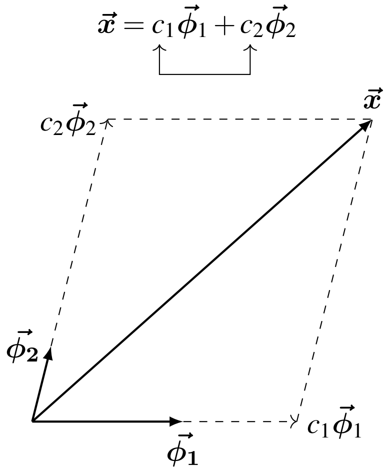
Definition 4.1 The ***dot product*** between two vectors $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is

$$<\vec{x}, \vec{y}> = \sum_{i=1}^{n} x_i y_i \tag{4.11}$$

The ***length*** of a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is

$$||\vec{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{<\vec{x}, \vec{x}>} \tag{4.12}$$

Q: Given a coordinate system $\vec{\phi}_1$, $\vec{\phi}_2$ and a vector $\vec{x}$, how can we express $\vec{x}$ as a weighted sum of $\vec{\phi}_1$ and $\vec{\phi}_2$?
i.e.



A: We "project" $\vec{x}$ on $\vec{\phi}_1$ and on $\vec{\phi}_2$, i.e. we take the dot product of $\vec{x}$ with $\vec{\phi}_1$ and $\vec{\phi}_2$ as folows:

$$\begin{aligned} <\vec{x},\vec{\phi}_1> &=< c_1\vec{\phi}_1 + c_2\vec{\phi}_2, \vec{\phi}_1 > \\ &= c_1 < \vec{\phi}_1, \vec{\phi}_1 > + c_2 < \vec{\phi}_2, \vec{\phi}_1 > \quad \text{(due to linearity of the dot product operator)} \end{aligned}$$

$<\vec{x},\vec{\phi}_1> =< \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} >= 7 \cdot 2 + 5 \cdot 1 = 19$, $<\vec{\phi}_1,\vec{\phi}_1> =< \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} >= 5$, and $<\vec{\phi}_2,\vec{\phi}_1> = < \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} >= 3$, so

$$19 = 5c_1 + 3c2 \tag{4.13}$$

Similarly, we "project" to $\vec{\phi}_2$

$$\begin{aligned} <\vec{x},\vec{\phi}_2> &=< c_1\vec{\phi}_1 + c_2\vec{\phi}_2, \vec{\phi}_2 > \\ &= c_1 < \vec{\phi}_1, \vec{\phi}_2 > + c_2 < \vec{\phi}_2, \vec{\phi}_2 > \end{aligned}$$

$<\vec{x},\vec{\phi}_2> =< \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 7 \cdot 1 + 5 \cdot 1 = 12$, $<\vec{\phi}_1,\vec{\phi}_2>= 3$, and $<\vec{\phi}_2,\vec{\phi}_2> =< \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 2$, so

$$12 = 3c_1 + 2c_2 \tag{4.14}$$

Solve the linear system of equations.

$$(4.13),\ (4.14) \implies c_1 = 2,\ c_2 = 3$$

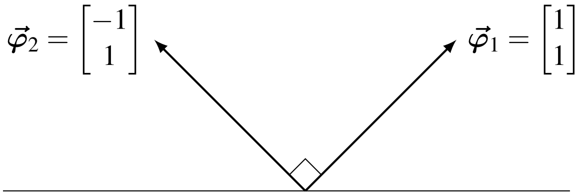
Observed that c_1, c_2 have to be found jointly through a solution of a 2×2 linear system. In general, for vectors in $\mathbb{R}^n$ we need to solve an $n \times n$ linear system!

Q: What about other coordinate systems?
A: There is one set of coordinates which is "better" (i.e. easier to work with) than others.

Definition 4.2 We say two vectors $\vec{x}$, $\vec{y}$ are ***orthogonal*** to each other (notation $\vec{x} \perp \vec{y}$) if

$$<\vec{x},\vec{y}>= 0 \tag{4.15}$$

E.g.



$$< \vec{\phi}_1, \vec{\phi}_2 >= (-1)(1) + 1(1) = 0 \text{ so } \vec{\phi}_1 \perp \vec{\phi}_2$$

Assume $\vec{\phi}_1 \perp \vec{\phi}_2$ and apply the previous procedure.

$$< \vec{x}, \vec{\phi}_1 > = c_1 < \vec{\phi}_1, \cancelto{\left|\left|\vec{\phi}_1\right|\right|^2}{\vec{\phi}_2} > + c_2 < \vec{\phi}_2, \cancelto{0 \text{ orthogonal}}{\vec{\boldsymbol{\phi}}_{\mathbf{1}}} >$$

$$\implies c_1 = \frac{< \vec{x}, \vec{\phi}_1 >}{\left|\left|\vec{\phi}_1\right|\right|^2}$$

Similarly,

$$< \vec{x}, \vec{\phi}_2 > = c_1 < \vec{\phi}_1, \cancelto{0}{\vec{\phi}_2} > + c_2 < \vec{\phi}_2, \cancelto{\left|\left|\vec{\phi}_1\right|\right|^2}{\vec{\boldsymbol{\phi}}_{\mathbf{2}}} >$$

$$\implies c_2 = \frac{< \vec{x}, \vec{\phi}_2 >}{\left|\left|\vec{\phi}_2\right|\right|^2}$$

So, **no need** to solve a linear system of equations, e.g. for $\vec{\phi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{\phi}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$,

$$c_1 = \frac{< \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >}{\left|\left|\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right|\right|^2} = \frac{7 \cdot 1 + 5 \cdot 1}{2} = 6$$

$$c_2 = \frac{< \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} >}{\left|\left|\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right|\right|^2} = \frac{-7 + 5}{2} = -1$$

so

$$\vec{x} = 6\vec{\phi}_1 - \vec{\phi}_2$$

In general in $\mathbb{R}^n$ if $\vec{\phi}_1$, $\vec{\phi}_2$, ..., $\vec{\phi}_n$ are orthogonal, then any vector $\vec{x}$ can be written as

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{\phi}_i \tag{4.16}$$

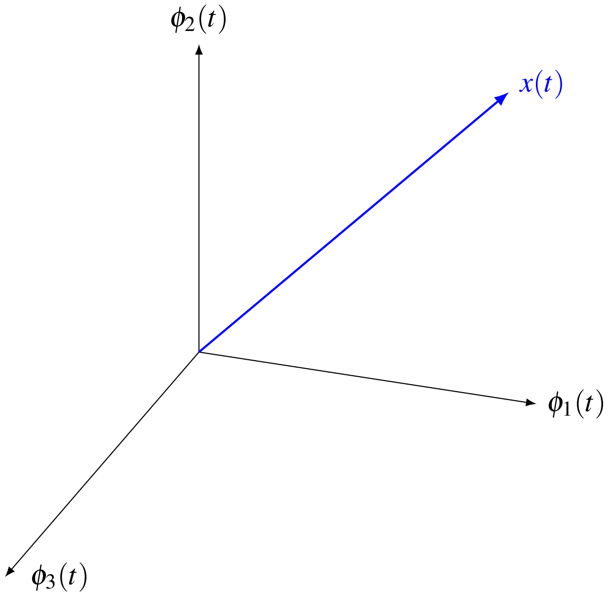
with

$$c_i = \frac{< \Sigma, \vec{\phi}_i >}{\left|\left|\vec{\phi}_i\right|\right|^2}, \; i = 1, 2, \ldots, n \tag{4.17}$$

Q: How are all these related to Fourier Series of periodic signals?
A: We can think of a periodic signal $x(t)$ as a "vector" in an ∞-dimensional space and the complex exponentials

$$\phi_n(t) = e^{jn\omega_0 t},\ n = 0, \pm 1, \pm 2, \ldots \tag{4.18}$$

as vectors defining the coordinate system.



Definition 4.3 We define the ***inner product*** (generalization of dot product) between signals as

$$< x(t), y(t) >= \int_0^{T_0} x(t)y^*(t)dt \tag{4.19}$$

Also,

$$||x(t)|| = \sqrt{< x(t), x(t) >} = \sqrt{\int_0^T |x(t)|^2 dt} = \underset{\substack{\uparrow \\ \text{energy of } x}}{\sqrt{E_x}} \tag{4.20}$$

We say two signals are orthogonal

$$x(t) \perp y(t) \iff < x(t), y(t) >= 0 \iff \int_0^T x(t)y^*(t)dt = 0 \text{ (as before)} \tag{4.21}$$

We would like to express as signal $x(t)$ as

$$x(t) = \sum_n c_n \underset{\substack{\uparrow \\ e^{jn\omega_0 t}}}{\phi_n(t)} \tag{4.22}$$

Fact:

$$\phi_n(t) \perp \phi_m(t),\ \forall n \neq m \tag{4.23}$$

Indeed,

$$< \phi_n(t), \phi_m(t) > = \int_0^T e^{jn\omega_0 t} \left(e^{jm\omega_0 t}\right)^* dt \tag{4.24}$$

$$= \int_0^{T_0} e^{j(n-m)\omega_0 t} dt = 0, \; n \neq m \tag{4.25}$$

Also,

$$||\phi_n(t)||^2 = \int_0^T \left|e^{jn\omega_0 t}\right|^2 dt = T \tag{4.26}$$

Following the same ideas as in the case of 2D vectors, since ϕ_n's are orthogonal we can write:

$$c_n = \frac{< x(t), \phi_n(t) >}{||\phi_n(t)||^2} = \frac{\int_0^T x(t) e^{-jn\omega_0 t} dt}{T}$$

$$= \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt \tag{4.27}$$

But this is exactly the Fourier Series coefficients!

4.7 Summary

Fourier Series is nothing more (nothing less) than a representation of a "vector" as a weighted sum of orthogonal "vectors"

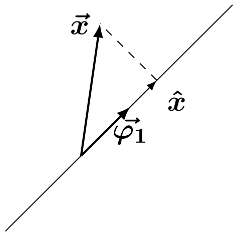
$$\Downarrow$$

Geometric Interpretation

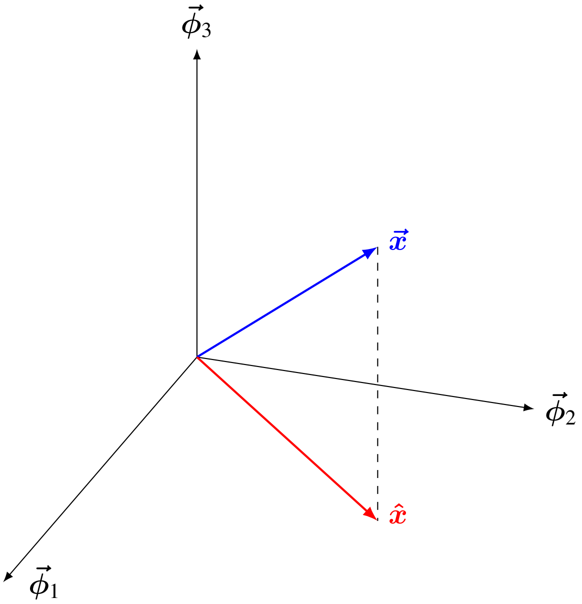
R We need to work in an ∞-dimensional space! (unlike $\mathbb{R}^n$ which is n-dimensional).

4.7.1 Approximation of Vectors

- What is the "best" approximation of $\vec{x}$ that lies in the line determined by $\vec{\phi}_n$?



- What is the "best" approximation of $\vec{x}$ that lies in the plane defined by $\vec{\phi}_1$, $\vec{\phi}_2$?

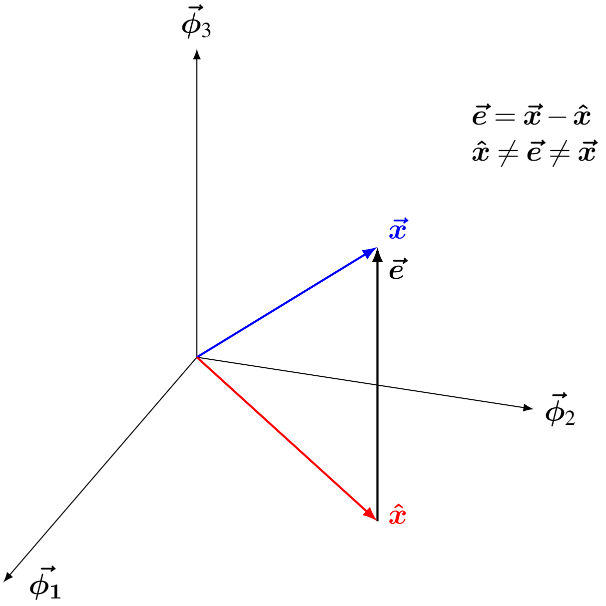


General statement: Given a vector $\vec{x} \in \mathbb{R}^n$ and a set of orthogonal vectors $\vec{\phi}_1, \vec{\phi}_2, \ldots, \vec{\phi}_N$ $(N < n)$, what is the "best" approximation of $\vec{x}$ in the form

$$\hat{x} = \sum_{i=1}^{N} \underset{\substack{\uparrow \\ \text{what are these coeff's?}}}{\hat{c}_i} \vec{\phi}_i$$

depends on what we mean with "best" approximation.
For us, "best" will mean that the approximation minimizes the "length" of the error.

$$\min_{\hat{c}_1,\ldots,\hat{c}_n} \| \underbrace{\vec{x} - \hat{x}}_{\vec{e}} \|$$



Proposition 4.4 The "best" approximation can be found by the orthogonal projection of $\vec{x}$ on the space spanned by ϕ_i's

$$\hat{c_i} = \frac{<\vec{x},\vec{\phi}_i>}{\left|\left|\vec{\phi}_i\right|\right|^2} \tag{4.28}$$

Proof.

$$\min_{\hat{c}} \underbrace{\left|\left|\vec{x}-\sum_i c_i\vec{\phi}_i\right|\right|^2}_{\downarrow}$$

$$\begin{aligned} J = \left|\left|\vec{x}-\sum_i c_i\vec{\phi}_i\right|\right|^2 &= ||x||^2 + \underbrace{\left|\left|\sum_i c_i\vec{\phi}_i\right|\right|^2}_{\downarrow} -2\sum_i c_i <\vec{x},\vec{\phi}_i> \\ &= \sum_i c_i^2\left|\left|\vec{\phi}_i\right|\right|^2 - 2\sum_i c_i <\vec{x},\vec{\phi}_i> \end{aligned} \tag{4.29}$$

To minimize[1]:

$$\begin{aligned} \frac{\partial J}{\partial c_i} = 0 &\iff \cancel{2}c_i\left|\left|\vec{\phi}_i\right|\right|^2 - \cancel{2} <\vec{x},\vec{\phi}_i> = 0 \\ &\iff c_i = \frac{<\vec{x},\vec{\phi}_i>}{\left|\left|\vec{\phi}_i\right|\right|^2} \end{aligned}$$

□

Q: Given an arbitrary signal $x(t)$ and a set of ***orthogonal*** functions $\phi_1(t),\dots,\phi_N(t)$ (N possibly ∞) ***over*** $[\alpha,\beta]$, how should we choose $\hat{c_1},\hat{c_2},\dots,\hat{c_N}$ such that $\sum_{i=1}^{N}\hat{c_i}\phi_i(t)$ is as "close" to $x(t)$ as possible over an interval $[\alpha,\beta]$?
A: It depends on what we mean by "close." One criterion is to change c_i's so that we minimize the

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[1] Or alternatively, "complete the squares" in Equation (4.29)

$$\sum_i\left[c_i^2\left|\left|\vec{\phi}_i\right|\right|^2 - 2c_i\left|\left|\vec{\phi}_i\right|\right| \cdot <\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}> + <\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}>^2 - <\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}>^2\right]$$

$$= \sum_i\left[\underbrace{\left(c_i\left|\left|\vec{\phi}_i\right|\right| - <\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}>\right)^2}_{\text{to minimize set to 0}} - <\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}>^2\right]$$

$$\Longrightarrow c_i = \frac{<\vec{x},\frac{\vec{\phi}_i|}{\left|\left|\vec{\phi}_i\right|\right|}>}{\left|\left|\vec{\phi}_i\right|\right|} \Longrightarrow c_i = \frac{<\vec{x},\vec{\phi}_i>}{\left|\left|\vec{\phi}_i\right|\right|^2}$$

integrated square error (ISE).

$$e(t) = x(t) - \sum_{i=1}^{N} \hat{c}_i \phi_i(t) \quad \text{(error)} \tag{4.30}$$

$$\text{ise} = \int_{\alpha}^{\beta} |e(t)|^2 dt \quad \text{(energy of the difference over } [a,b]\text{)} \tag{4.31}$$

Proposition 4.5 **Fact:** The ISE is minimized by

$$\hat{c}_i = \frac{< x(t), \phi_i(t) >}{||\phi_i(t)||^2} \tag{4.32}$$

Proof.

$$
\begin{aligned}
\mathrm{ISE} &= \int_\alpha^\beta |e(t)|^2 dt \\
&= \int_\alpha^\beta \left| x(t) - \sum_{i=1}^N \hat{c}_i \phi_i(t) \right|^2 dt \\
&= \underbrace{\int_\alpha^\beta |x(t)|^2 dt}_{C} + \underbrace{\int_\alpha^\beta \left| \sum_{i=1}^N \hat{c}_i \phi_i(t) \right|^2 dt}_{A} \\
&\left.\begin{array}{l} [1] - \displaystyle\int_\alpha^\beta x(t) \left(\sum_{i=1}^N \hat{c}_i{}^* \phi_i^*(t) \right) dt \\ \quad - \displaystyle\int_\alpha^\beta \left(\sum_{i=1}^N \hat{c}_i \phi_i(t) \right) x^*(t) dt \end{array}\right\} B
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_\alpha^\beta \left[\sum_{i=1}^N \hat{c}_i \phi_i(t) \right] \left[\sum_{j=1}^N \hat{c}_j \phi_j(t) \right]^* dt \\
&= \sum_i \sum_j \hat{c}_i \hat{c}_j \int_\alpha^\beta \phi_i(t) \phi_j^*(t) dt \\
&\overset{\text{(orthogonality)}}{=} \sum_i |\hat{c}_i|^2 \, ||\phi_i(t)||^2
\end{aligned}
$$

$$
\begin{aligned}
B &= -2\,\mathrm{Re} \left\{ \int_\alpha^\beta \left(\sum_{i=1}^N \hat{c}_i \phi_i(t) \right) x^*(t) dt \right\} \\
&= \sum_{i=1}^N \mathrm{Re} \left\{ -2\hat{c}_i \int_\alpha^\beta \phi_i(t) x^*(t) dt \right\} = \sum_{i=1}^n \mathrm{Re}\left\{ -2\hat{c}_i < \phi_i(t), x(t) > \right\}
\end{aligned}
$$

C does not depend on $\hat{c}_i$'s

$$
A + B = \sum_i \left\{ |\hat{c}_i|^2 \, ||\phi_i(t)||^2 - 2\,\mathrm{Re}\left\{ \hat{c}_i < \phi_i(t), x(t) > \right\} \right\}
$$

$$
\{\} = 2\,||\phi_i(t)||^2 - 2\,\mathrm{Re}\left\{ \underbrace{\hat{c}_i \, ||\phi_i(t)||} < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) > \right\} + \underset{\substack{\uparrow \\ \text{complete the square}}}{\left| < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) >^* \right|^2}
$$

$$
- \underset{\substack{\uparrow \\ \text{does not depend on } c_i\text{'s}}}{\left| < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) >^* \right|^2}
$$

$$
= \left(\hat{c}_i \, ||\phi_i(t)|| - < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) >^* \right)^2 - \left| < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) > \right|^2
$$

To minimize, set

$$\begin{aligned}\hat{c_i} &= \frac{1}{||\phi_i(t)||} < \frac{\phi_i(t)}{||\phi_i(t)||}, x(t) >^* \\ &= \frac{1}{||\phi_i(t)||^2} < \phi_i(t), x(t) >^* \\ &= \frac{1}{E_i}\int_{\alpha}^{\beta} \alpha(t)\phi_i^*(t)dt\end{aligned}$$

And for $[\alpha,\beta] = [0,T]$ and $\phi_n(t) = e^{jn\omega_0 t}$ we get $\hat{c_n} = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$, i.e. EXACTLY the Fourier Series coefficients. □

Summary: The "best" (in terms of minimizing the ISE) approximation of a signal using a finite number of complex exponentials is given again by a finite (truncated) Fourier Series!

5. Fourier Transform

5.1 Introduction to Fourier Transform and Derivation

We have seen that any* periodic signal $x(t)$ with period T_0 can be represented as a Fourier Series (a weighted sum of complex exponentials)

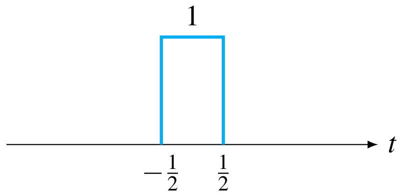
$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T_0}$$

with

$$c_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt$$

The ***Fourier Transform*** (FT) is the extension of this idea to non-periodic signals. We will "derive" the Fourier Transform as a limit of the Fourier Series of a periodic signal as $T_0 \to \infty$.

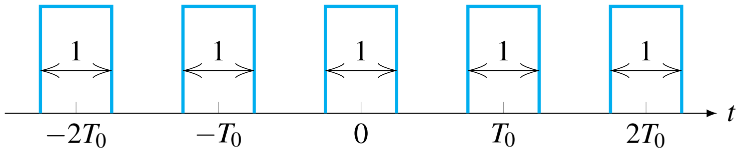
Consider a non-periodic signal $x(t)$ (e.g. $x(t) = \text{rect}(t)$)



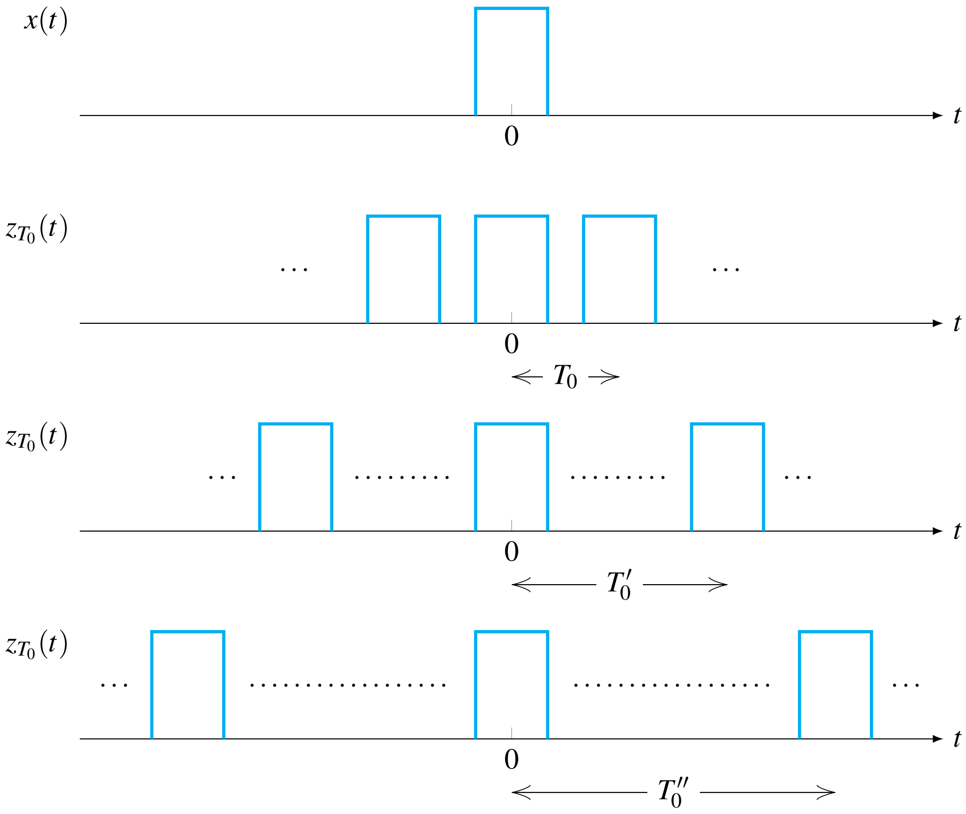
Create a periodic extension of this signal with period T_0

$$z_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \underset{\substack{\uparrow \\ \text{delayed version of } x(t) \\ \text{by } nT_0}}{x(t-nT_0)}$$

$z_{T_0}(t) =$



Clearly $z_{T_0}(t)$ is periodic. Also, as $T_0 \to \infty$, $z_{T_0}(t)$ "resembles" more and more $x(t)$ over a longer range of values of t

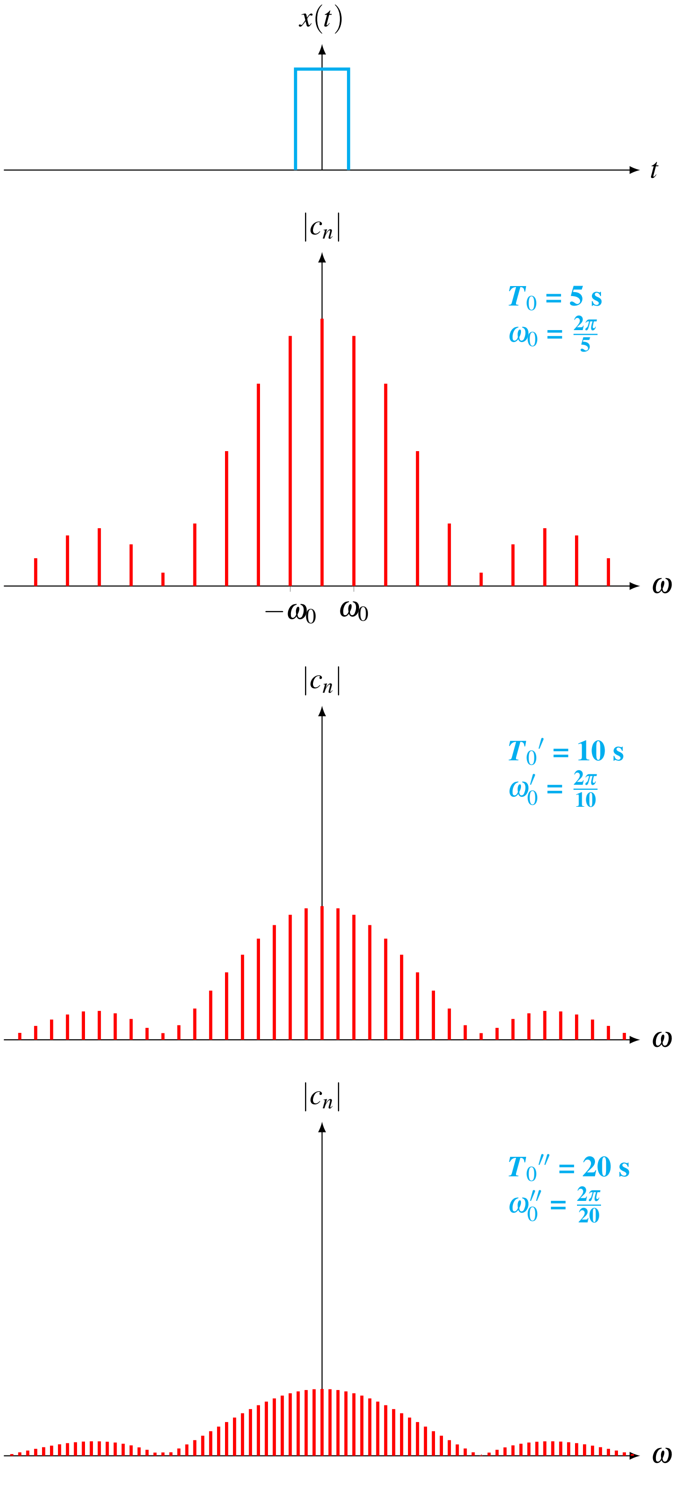


Let's examine the Fourier Series of $z_{T_0}(t)$

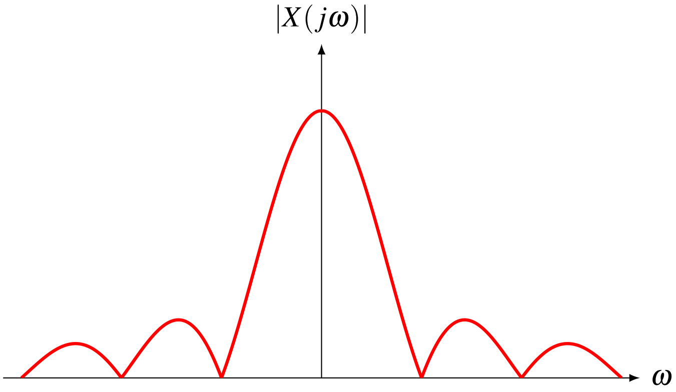
$$z_{T_0}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T_0}$$
$$c_n = \frac{1}{T_0} \int_0^{T_0} z_{T_0}(t) e^{-jn\omega_0 t}$$

In our example:

$$c_n = \frac{1}{T_0} \operatorname{sinc}\left(\frac{n\omega_0}{2}\right)$$



In the limit, as $T_0 \to \infty$, $z_{T_0}(t) \to x(t)$, and the spectrum of $z_{T_0}(t)$ becomese a continuous function of ω (call it $X(j\omega)$).



See MATLAB demo.

$$c_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z_{T_0}(t)e^{-jn\omega_0 t}dt \tag{5.1}$$

$$z_{T_0}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \tag{5.2}$$

As

$$\left|\begin{array}{l} T_0 \to \infty \\ \omega_0 = \frac{2\pi}{T_0} \to 0 \\ z_{T_0}(t) \to x(t) \\ n\omega_0 \to \omega \end{array}\right. \quad \text{(line spectrum becomes continuous)}$$

$$(8.2) \implies c_n T_0 = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z_{T_0}(t)e^{-jn\omega_0 t} \to \underbrace{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}_{\triangleq X(j\omega)}$$

Also,

$$\begin{aligned}(8.3) \implies z_{T_0}(t) &= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty}(c_n T_0)e^{jn\omega_0 t} \\ &= \underbrace{\frac{\omega_0}{2\pi}\sum_{n=-\infty}^{+\infty}}_{\substack{\downarrow \\ \frac{1}{2\pi}\int d\omega}} \underbrace{\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z_{T_0}(t)e^{-jn\omega_0 t}dt\right]}_{X(j\omega)} \underbrace{e^{jn\omega_0 t}}_{\substack{\downarrow \\ e^{j\omega t}}}\end{aligned}$$

so in the limit as $T_0 \to \infty$ we get

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Definition 5.1 — Fourier Transform. We say that

$$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{5.3}$$

is the ***Fourier Transform*** of $x(t)$ and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \tag{5.4}$$

Notation:

$$\underbrace{x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)}_{\text{FT pair}}$$

$$X(j\omega) = \mathscr{F}\{x(t)\}, x(t) = \mathscr{F}^{-1}\{X(j\omega)\}$$

R

- A sufficient condition for existence of $X(j\omega)$ is that
  i) $x(t)$ is ***absolutely integrable***, i.e.
  $$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$
  ii) $x(t)$ is piecewise continuous
- Furthermore, if $x(t)$ is differentiable at all points of continuity and the left/right derivatives exist at all points then $X(j\omega)$ is continuous and the inverse transform exists for all t.

5.1.1 Interpretation

Think of integral as a sum:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ &\approx \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{+\infty} X(jn\Delta\omega) \underbrace{e^{j(n\Delta\omega)t}}_{\substack{\Uparrow \\ \text{complex} \\ \text{exponential} \\ \text{with freq} \\ n\cdot\Delta\omega}} \end{aligned}$$

- Every signal is a weighted sum of complex exponentials
- Unlike periodic signals where only the harmonics $n\omega_0$ of the fundamental frequency are entering in the description, here we need complex exponentials of all possible frequencies $\omega \in (-\infty, +\infty)$.

R A note about $\frac{1}{2\pi}$ factor in the inverse Fourier Transform:
If instead of angular frequency ω we use natural frequency $f = \frac{\omega}{2\pi}$ then we can redefine

$$\hat{X}(jf) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \tag{5.5}$$

$$\begin{aligned} x(t) &= \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \hat{X}(jf) e^{j2\pi f t} \cancel{2\pi} df \\ &= \int_{-\infty}^{\infty} \hat{X}(jf) e^{j2\pi f t} df \qquad (5.6) \end{aligned}$$

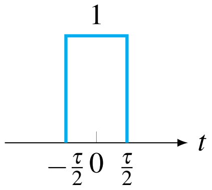
so the two equations are symmetric.

5.2 Examples of Fourier Transform Pairs

1.

$$\boxed{\operatorname{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathscr{F}} \tau \cdot \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)} \tag{5.7}$$

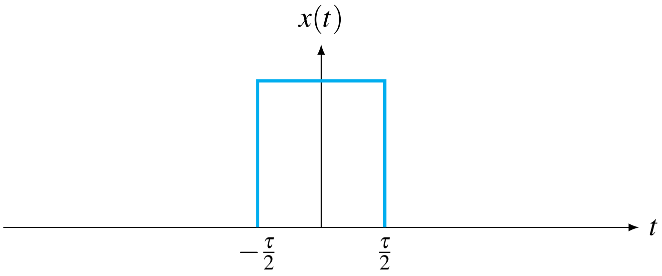
Proof. $x(t) = \operatorname{rect}\left(\frac{t}{\tau}\right)$

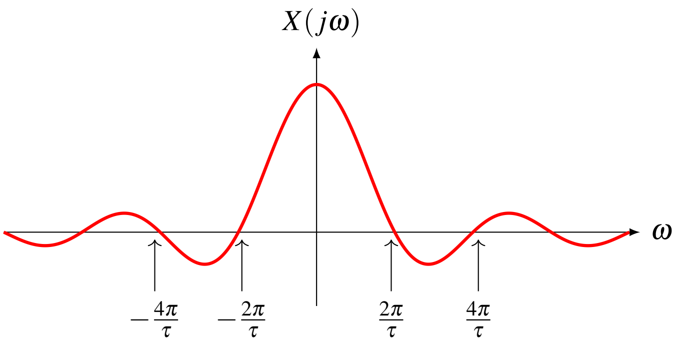


$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
&= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} (1) \cdot e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{e^{-j\omega\frac{\tau}{2}} - e^{+j\omega\frac{\tau}{2}}}{-j\omega} = \frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j\omega} 2 \\
&= \frac{\cancel{2}\sin\left(\frac{\omega\tau}{2}\right)}{\omega} \frac{\tau/\cancel{2}}{\frac{\tau}{2}} = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \\
&= \tau \operatorname{Sa}\left(\frac{\omega\tau}{2}\right) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)
\end{aligned}$$

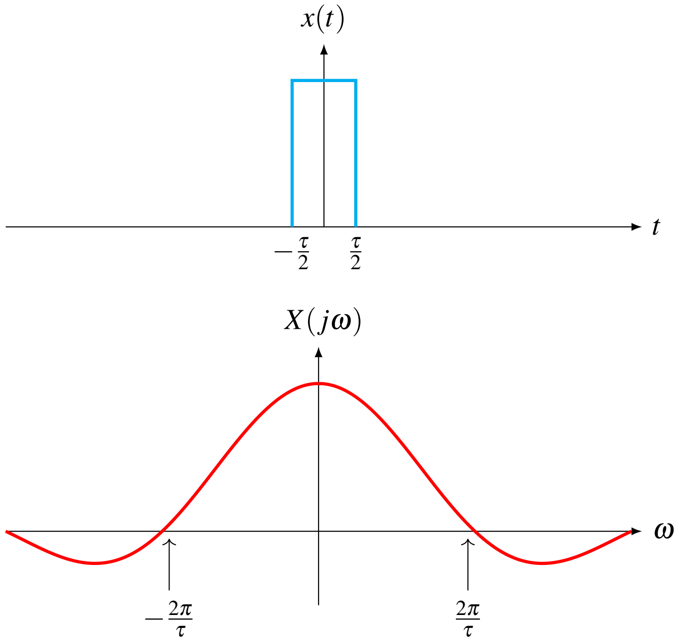
□

Effect of τ:





Suppose we decrease τ, which means a narrower pulse in time.

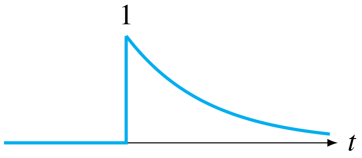


The Fourier Transform spreads! (uncertainty principle!)

2.

$$\boxed{x(t)=e^{-\alpha t}u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{\alpha+j\omega},\ \forall\alpha>0} \tag{5.8}$$

Proof. $x(t)=e^{-at}u(t)$, $\alpha>0$

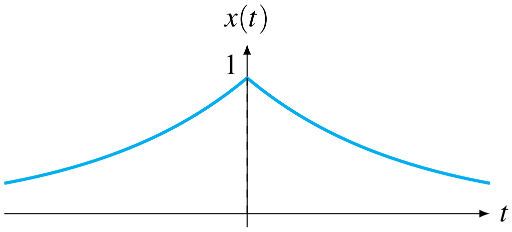


$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{+\infty} e^{-\alpha t}e^{-j\omega t}dt \\ &= \int_{0}^{+\infty} e^{-(\alpha+j\omega)t}dt = \left.\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right|_{0}^{+\infty} \\ &= 0-\frac{1}{-(\alpha+j\omega)} = \frac{1}{\alpha+j\omega} \end{aligned}$$

This result is true even if α is a complex number and $\mathrm{Re}\{\alpha\}>0$. $\square$

3.

$$\boxed{x(t)=e^{-\alpha|t|} \overset{\mathscr{F}}{\longleftrightarrow} \frac{2\alpha}{\alpha^2+\omega^2},\ \forall\alpha>0} \tag{5.9}$$



Proof.

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-\alpha|t|}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{-\alpha(-t)}e^{-j\omega t}dt + \int_{0}^{+\infty} e^{-\alpha t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(\alpha-j\omega)t}dt + \int_{0}^{+\infty} e^{-(\alpha+j\omega)t}dt \\
&= \left.\frac{e^{(\alpha-j\omega)t}}{\alpha-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right|_{0}^{+\infty} \\
&= \frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} = \frac{2\alpha}{\alpha^2+\omega^2}
\end{aligned}
$$

This result also holds for complex values of α, with $\mathrm{Re}\{\alpha\} > 0$. □

4.

$$\boxed{x(t) = te^{-\alpha t}u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{(\alpha+j\omega)^2},\ \mathrm{Re}\{\alpha\} > 0} \tag{5.10}$$

Proof. Let's do a trick: We know

$$
\begin{aligned}
e^{-\alpha t}u(t) &\overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{\alpha+j\omega} \\
&\Updownarrow \\
\frac{1}{\alpha+j\omega} &= \int_{-\infty}^{\infty} e^{-\alpha t}u(t)e^{-j\omega t}dt
\end{aligned}
$$

Take the derivative of both sides with respect to α, i.e. $\frac{d}{d\alpha}$:

$$
\begin{aligned}
\frac{d}{d\alpha}\left(\frac{1}{\alpha+j\omega}\right) &= \int_{-\infty}^{\infty} -te^{-\alpha t}u(t)e^{-j\omega t}dt \\
\Longleftrightarrow\ -\frac{d}{d\alpha}\left(\frac{1}{\alpha+j\omega}\right) &= \underbrace{\int_{-\infty}^{+\infty} te^{-\alpha t}u(t)e^{-j\omega t}dt}_{X(j\omega)}
\end{aligned}
$$

So

$$X(j\omega) = -\frac{-1}{(\alpha+j\omega)^2} = \frac{1}{(\alpha+j\omega)^2}$$

□

5.

$$\boxed{\delta(t) \overset{\mathscr{F}}{\longleftrightarrow} 1} \tag{5.11}$$

Proof.

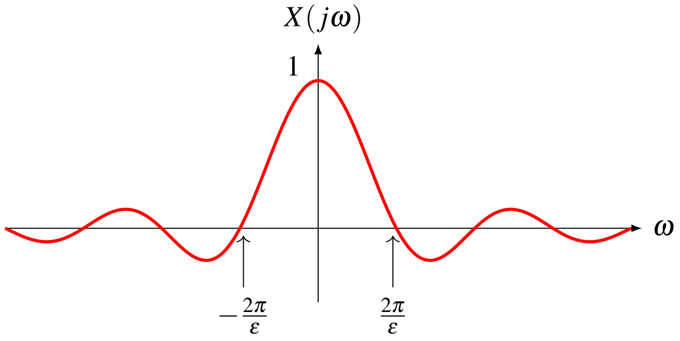
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\big|_{t=0} = 1$$

Alternative derivation: Consider $x_\varepsilon(t) = \frac{1}{\varepsilon}\operatorname{rect}\left(\frac{t}{\varepsilon}\right)$. Recall that

$$\operatorname{rect}\left(\frac{t}{\varepsilon}\right) \overset{\mathscr{F}}{\longleftrightarrow} \varepsilon \operatorname{sinc}\left(\frac{\omega\varepsilon}{2\pi}\right)$$

so

$$\begin{array}{ccc} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) & \overset{\mathscr{F}}{\longleftrightarrow} & \varepsilon \operatorname{sinc}\left(\frac{\omega\varepsilon}{2\pi}\right) \\ \downarrow \varepsilon \to 0 & & \downarrow \varepsilon \to 0 \\ \delta(t) & \overset{\mathscr{F}}{\longleftrightarrow} & ? \end{array}$$



$\lim_{\varepsilon \to 0} \operatorname{sinc}\left(\frac{\omega\varepsilon}{2\pi}\right) = 1$, so

$$\delta(t) \overset{\mathscr{F}}{\longleftrightarrow} 1$$

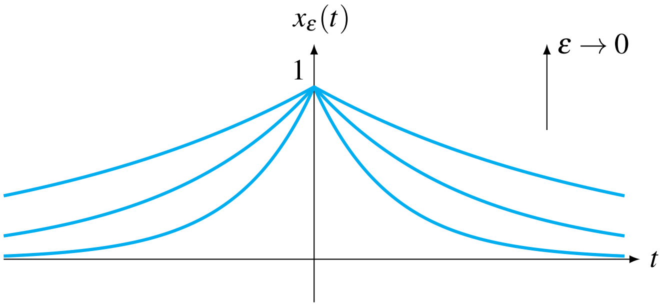
□

So far we have seen examples where $x(t)$ satisfies the sufficient conditions stated earlier for the existence of Fourier Transform. But we are also interested in the Fourier Transform of other functions such as $\delta(t)$, $u(t)$, $e^{j\omega_0 t}$, and $\cos(\omega_0 t)$ which are not absolutely integrable. It is common to extend the theory formally to include these functions.

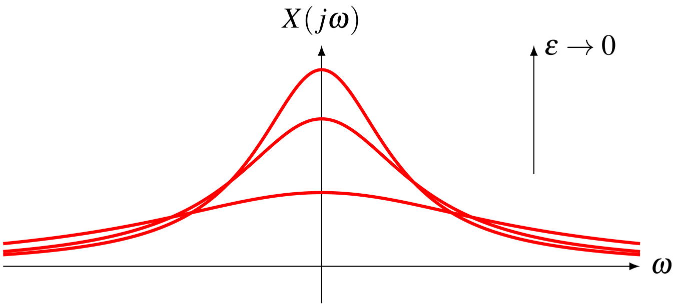
6.

$$\boxed{x(t) = 1 \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega)} \tag{5.12}$$

Proof. $x(t)$ is not absolutely integrable, so consider $x_\varepsilon(t) = e^{-\varepsilon|t|} \xrightarrow{\varepsilon \to 0} 1$.



$$\begin{tikzcd} x_\varepsilon(t) = e^{-\varepsilon|t|} \arrow[r, leftrightarrow, "\mathscr{F}"] \arrow[d, "\varepsilon \to 0"] & X_\varepsilon(j\omega) \overset{(\text{ex.}3)}{=} \dfrac{2\varepsilon}{\varepsilon^2+\omega^2} \arrow[d, "\varepsilon \to 0"] \\ 1 \arrow[r, leftrightarrow, "\mathscr{F}"] & ? \end{tikzcd}$$



As it turns out the family of functions $X_\varepsilon(j\omega) = \frac{2\varepsilon}{\varepsilon^2+\omega^2}$ are

i) even

ii) have area = 2π

iii) $\lim_{\varepsilon\to 0} X_\varepsilon(j\omega) = 0,\ \omega \neq 0$

iv) $\lim_{\varepsilon\to 0} X_\varepsilon(j0) = \infty,\ \omega = 0$

Therefore,

$$X(j\omega) = \lim_{\varepsilon\to 0} X_\varepsilon(j\omega) = 2\pi\delta(\omega)$$

☐

What is the interpretation of this result?

- The signal $x(t) = 1$ is constant, so it only has a DC component. That's why $X(j\omega) = 0,\ \forall\omega \neq 0$ (for all $\omega \neq 0$) except at $\omega = 0$ (DC term)!

7.

$$\boxed{e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\delta(\omega - \omega_0)} \tag{5.13}$$

Proof. Since $|x(t)| = \left|e^{j\omega_0 t}\right| = 1$, it is not absolutely integrable, so we can think of $x_\varepsilon(t) = e^{j\omega_0 t}e^{-\varepsilon|t|} \xrightarrow[\varepsilon\to 0]{} e^{j\omega_0 t}$

$$\begin{aligned} X_\varepsilon(j\omega) = \mathscr{F}\{x_\varepsilon(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-\varepsilon|t|}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-\varepsilon|t|}e^{-j(\omega-\omega_0)t}dt \\ &\overset{\text{(ex. 3)}}{=} \frac{2\varepsilon}{\varepsilon^2+(\omega-\omega_0)^2} \xrightarrow{\varepsilon\to 0} 2\pi\delta(\omega-\omega_0) \end{aligned}$$

□

Interpretation: $e^{j\omega_0 t}$ is a complex exponential at frequency ω_0 so $X(j\omega)$ is 0 $\forall\omega \neq \omega_0$ (for all $\omega \neq \omega_0$).

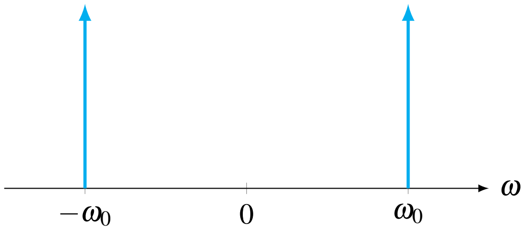
8.

$$\boxed{\begin{aligned} \cos(\omega_0 t) &\overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0) && (5.14a)\\ \sin(\omega_0 t) &\overset{\mathscr{F}}{\longleftrightarrow} \frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0) && (5.14b) \end{aligned}}$$

Proof. We will see later that $\mathscr{F}\{\cdot\}$ is a linear operator, i.e.

$$\begin{aligned} \mathscr{F}\{\alpha x_1(t)+\beta x_2(t)\} &= \alpha\mathscr{F}\{x_1(t)\}+\beta\mathscr{F}\{x_2(t)\} \\ \cos(\omega_0 t) &= \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} \\ \mathscr{F}\{\cos(\omega_0 t)\} &= \frac{1}{2}\mathscr{F}\left\{e^{j\omega_0 t}\right\}+\frac{1}{2}\mathscr{F}\left\{e^{-j\omega_0 t}\right\} \\ &= \frac{1}{\cancel{2}}\cancel{2}\pi\delta(\omega-\omega_0)+\frac{1}{\cancel{2}}\cancel{2}\pi\delta(\omega+\omega_0) \end{aligned}$$

and similarly for $\sin(\omega_0 t)$.



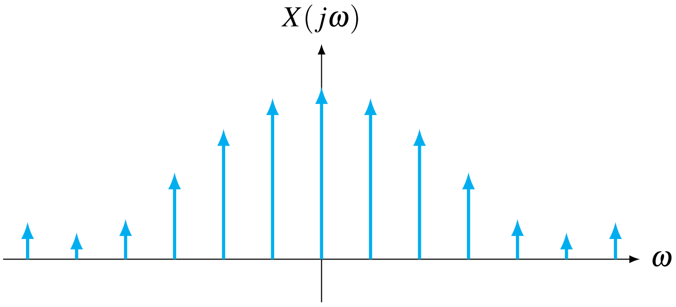
□

9. Consider a periodic function $x(t)$ with Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

Then, using linearity of Fourier Transform, we have

$$\begin{aligned} X(j\omega) = \mathscr{F}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right\} &= \sum_{n=-\infty}^{+\infty} c_n\mathscr{F}\left\{e^{jn\omega_0 t}\right\} \\ &= \sum_{n=-\infty}^{+\infty} c_n\delta(\omega-n\omega_0) \qquad (5.15) \end{aligned}$$



10.

$$\boxed{\operatorname{sgn}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}} \tag{5.16}$$

Proof. Again, since $|\operatorname{sgn}(t)| = 1$, this is NOT an absolutely integrable signal, so consider $x_\varepsilon(t) = \operatorname{sgn}(t) \cdot e^{-\varepsilon|t|}$.

$$X_\varepsilon(j\omega) = \cdots = -\frac{2j\omega}{\varepsilon^2 + \omega^2}$$

so

$$X(j\omega) = \lim_{\varepsilon \to 0} X_\varepsilon(j\omega) = -\frac{2j\cancel{\omega}}{\omega^{\cancel{2}}} = \frac{2}{j\omega}$$

☐

11.

$$\boxed{u(t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}} \tag{5.17}$$

Proof.

$$u(t) = \frac{1}{2}\operatorname{sgn}(t) + \frac{1}{2} \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{\cancel{2}}\frac{\cancel{2}}{j\omega} + \frac{1}{\cancel{2}}2\pi\delta(\omega)$$

☐

Although we have been quite careless in deriving these results (interchanging limits and integrals), the Fourier Transform pairs give the correct results when you apply them to actual problems.

5.3 Properties of the Fourier Transform

5.3.1 Linearity

If

$$x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(j\omega)$$
$$x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_2(j\omega)$$

then for all constants α and β,

$$\boxed{\alpha x_1(t) + \beta x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} \alpha X_1(j\omega) + \beta X_2(j\omega)} \tag{5.18}$$

$x(t) \xleftrightarrow{\text{F.T.}} X(j\omega)$		
		Eq.
$e^{-\alpha t}u(t)$, $\text{Re}\{\alpha\} > 0$	$\frac{1}{\alpha+j\omega}$	(5.8)
$e^{-\alpha\|t\|}$, $\text{Re}\{\alpha\} > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$	(5.9)
$\text{rect}\left(\frac{t}{\tau}\right)$	$\frac{\tau\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$	(5.7)
$te^{-\alpha t}u(t)$, $\text{Re}\{\alpha\} > 0$	$\frac{1}{(\alpha+j\omega)^2}$	(5.10)
$t^{n-1}e^{-\alpha t}u(t)$, $\text{Re}\{\alpha\} > 0$ & $n \geq 1$	$\frac{(n-1)!}{(\alpha+j\omega)^n}$	
$e^{-\alpha t^2}$, $\alpha > 0$ & real-valued	$\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	
1	$2\pi\delta(\omega)$	(5.12)
$\delta(t)$	1	(5.11)
$e^{\pm j\omega_0 t}$, ω_0 real-valued	$2\pi\delta(\omega \mp \omega_0)$	(5.13)
$\cos\omega_0 t$, ω_0 real-valued	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$	(5.14a)
$\sin\omega_0 t$, ω_0 real-valued	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$	(5.14b)
$\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$, ω_0 real-valued	$2\pi\sum\limits_{n=-\infty}^{\infty} c_n\delta(\omega-n\omega_0)$	(5.15)
$\text{sgn}(t)$	$\frac{2}{j\omega}$	(5.16)
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	(5.17)

Table 5.1: Common Fourier Transform pairs.

Proof.

$$\begin{aligned}\mathscr{F}\{\alpha x_1(t)+\beta x_2(t)\} &= \int_{-\infty}^{\infty}[\alpha x_1(t)+\beta x_2(t)]\,e^{-j\omega t}dt \\ &= \alpha\int_{-\infty}^{\infty}x_1(t)e^{-j\omega t}dt+\beta\int_{-\infty}^{\infty}x_2e^{-j\omega t}dt \\ &= \alpha X_1(j\omega)+\beta X_2(j\omega)\end{aligned}$$

□

■ Example 5.1

$$\begin{aligned} e^{-t}u(t) &\xleftrightarrow{\mathscr{F}} \frac{1}{1+j\omega} \\ e^{-2t}u(t) &\xleftrightarrow{\mathscr{F}} \frac{1}{2+j\omega} \\ \therefore\ 2e^{-t}u(t)+3e^{-2t}u(t) &\xleftrightarrow{\mathscr{F}} \frac{2}{1+j\omega}+\frac{3}{2+j\omega}\end{aligned}$$

■

5.3.2 Symmetry

$$\boxed{\begin{array}{ccc}\text{if} & x(t)\xleftrightarrow{\mathscr{F}}X(j\omega) & \text{then} \\ & x^*(t)\xleftrightarrow{\mathscr{F}}X^*(-j\omega) & \end{array}} \tag{5.19}$$

Proof. $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$ means

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$\Downarrow$$
$$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$
$$\Downarrow$$
$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt \overset{\text{def}}{=} \mathscr{F}\{x^*(t)\}$$

so

$$x^*(t) \overset{\mathscr{F}}{\longleftrightarrow} X^*(-j\omega)$$

□

Special case:

$$\begin{tikzcd}
 & \boxed{\begin{array}{c}\text{if } x(t) \text{ is real then}\\ X^*(-j\omega) = X(j\omega)\end{array}} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}|X(-j\omega)| = |X(j\omega)|\\ \text{(even)}\end{array} & & \begin{array}{c}\angle X(-j\omega) = -\angle X(j\omega)\\ \text{(odd)}\end{array}
\end{tikzcd}$$

5.3.3 Duality

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then

$$\boxed{X(jt) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi x(-\omega)} \tag{5.20}$$

Proof.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jz)e^{jzt}dz$$
$$\therefore\ 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jz)e^{-jz\omega}dz$$
$$\therefore\ 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt = \mathscr{F}\{X(jt)\}$$

□

■ Example 5.2

$$x(t) = e^{-\alpha|t|},\ \alpha \text{ real} \overset{\mathscr{F}}{\longleftrightarrow} \frac{2\alpha}{\alpha^2+\omega^2} = X(j\omega) \quad (5.9)$$
$$\therefore\ \frac{2\alpha}{\alpha^2+t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi e^{-\alpha|\omega|}$$

■

■ **Example 5.3**

$$x(t) = 1 \xleftrightarrow{\mathscr{F}} X(j\omega) = 2\pi\delta(\omega)$$

Then,

$$X(jt) = \cancel{2\pi}\delta(t) \xleftrightarrow{\mathscr{F}} \cancel{2\pi}x(-\omega) = 2\pi(1)$$
$$\Downarrow$$
$$\delta(t) \xleftrightarrow{\mathscr{F}} 1$$

(as we have already seen earlier). ■

■ **Example 5.4**

$$x(t) = \operatorname{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathscr{F}} X(j\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \tau \operatorname{Sa}\left(\frac{\omega\tau}{2}\right)$$

Then,

$$X(jt) = \tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right) \xleftrightarrow{\mathscr{F}} 2\pi x(-\omega) = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right)$$

If we let $\omega_0 = \frac{\tau}{2}$, then

$$\operatorname{sinc}\left(\frac{\omega_0 t}{\pi}\right) \xleftrightarrow{\mathscr{F}} \frac{\pi}{\omega_0} \operatorname{rect}\left(\frac{\omega}{2\omega_0}\right)$$

■

5.3.4 Time Shifting

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)$$

then

$$\boxed{x(t-t_0) \xleftrightarrow{\mathscr{F}} X(j\omega)e^{-j\omega t_0}} \tag{5.21}$$

for all real-valued t_0.

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt &= \int_{-\infty}^{\infty} x(\eta)e^{-j\omega(\eta+t_0)}d\eta \ (\text{let } \eta \equiv t-t_0) \\ &= e^{-j\omega t_0}\int_{-\infty}^{+\infty} x(\eta)e^{-j\omega\eta}d\eta \\ &= e^{-j\omega t_0}X(j\omega)\end{aligned}$$

☐

■ **Example 5.5**

$$e^{-\alpha t}u(t),\ \operatorname{Re}\{\alpha\} > 0 \xleftrightarrow{\mathscr{F}} \frac{1}{\alpha + j\omega}$$

$$\therefore e^{-\alpha(t-t_0)}u(t-t_0),\ \operatorname{Re}\{\alpha\} > 0 \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0}\frac{1}{\alpha + j\omega}$$

R Since $\delta(t) \xleftrightarrow{\mathscr{F}} 1$, we must have $\delta(t-t_0) \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0}$

■

5.3.5 Time Scaling

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then for any real-valued constant α,

$$\boxed{x(\alpha t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|\alpha|} X\left(\frac{j\omega}{\alpha}\right)} \tag{5.22}$$

Proof. (for the case where $\alpha > 0$. The case when $\alpha < 0$ is handled in a similar manner)

$$\begin{aligned} \int_{-\infty}^{\infty} x(\alpha t) e^{-j\omega t} dt &= \frac{1}{\alpha} \int_{-\infty}^{\infty} X(\eta) e^{-j\left(\frac{\omega}{\alpha}\right)\eta} d\eta \text{ (make the substitution } \eta \equiv \alpha t) \\ &= \frac{1}{\alpha} X\left(\frac{j\omega}{\alpha}\right) \end{aligned}$$

☐

> Thus expansion in the time-domain yields compression in the frequency domain, and compression in the time-domnain yields expansion in the frequency-domain.

⊛ Uncertainty Principle

5.3.6 Frequency Shifting and Modulation

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then for any real-valed ω_0,

$$\boxed{x(t) e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} X\left(j\left(\omega - \omega_0\right)\right)} \tag{5.23}$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} \left[x(t) e^{j\omega_0 t}\right] e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt \\ &= X\left(j\left(\omega - \omega_0\right)\right) \end{aligned}$$

☐

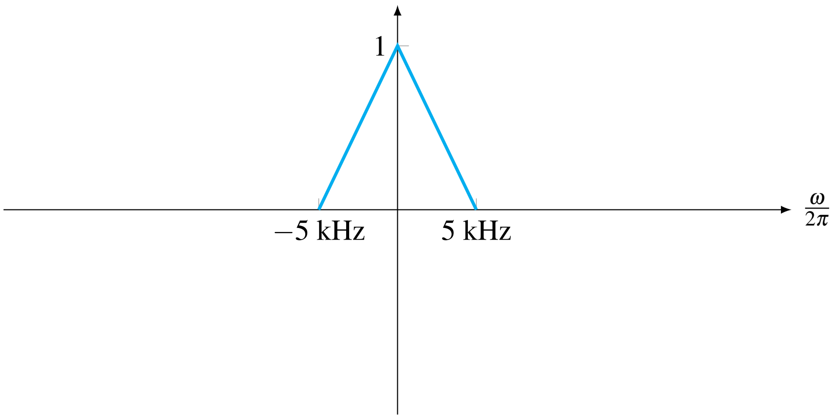
Thus, using the facts that $\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$ and $\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$, we conclude that if

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

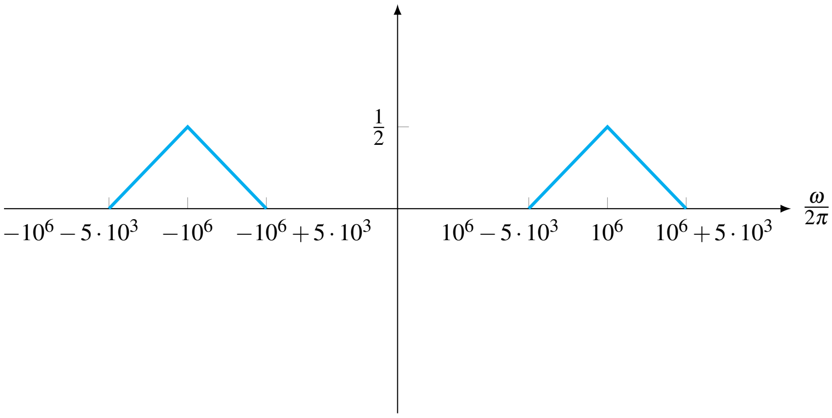
then

$$\boxed{\begin{aligned} x(t) \cos \omega_0 t &\overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{2}\left[X\left(j\left(\omega - \omega_0\right)\right) + X\left(j\left(\omega + \omega_0\right)\right)\right] && (5.24) \\ x(t) \sin \omega_0 t &\overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{2j}\left[X\left(j\left(\omega - \omega_0\right)\right) - X\left(j\left(\omega + \omega_0\right)\right)\right] && (5.25) \end{aligned}}$$

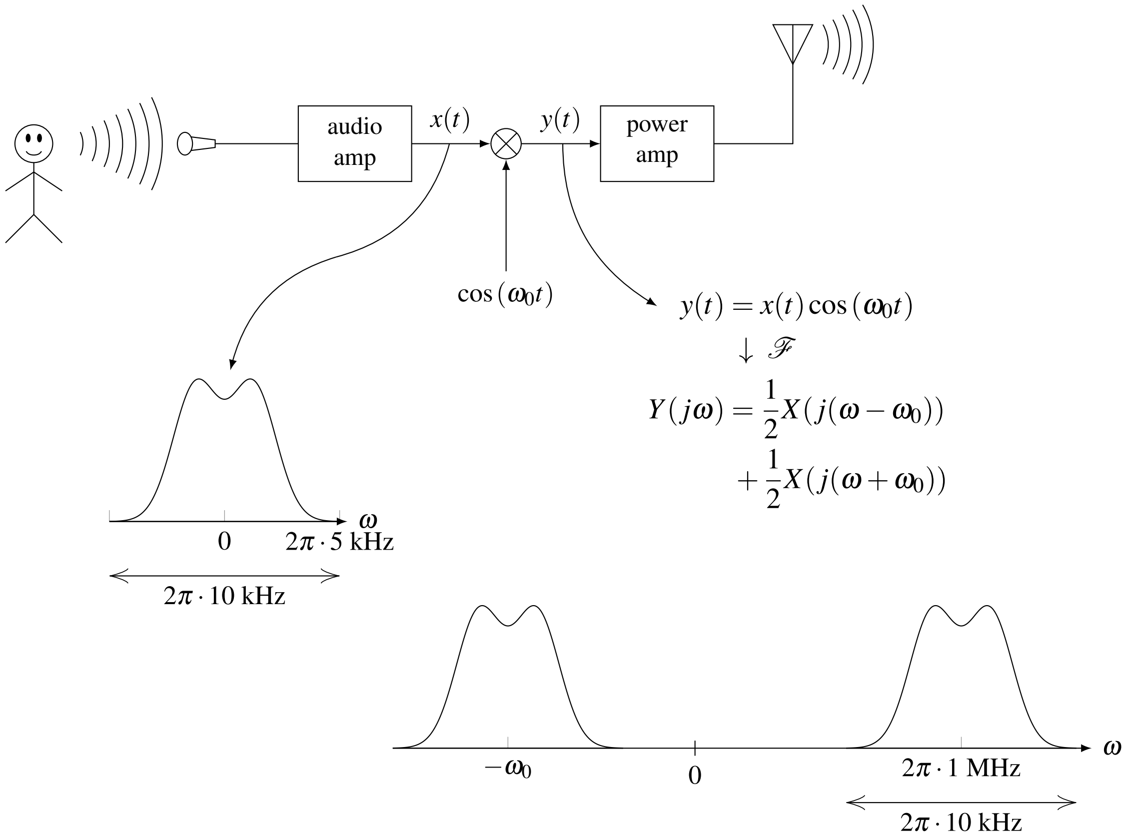
■ **Example 5.6** Suppose $x(t)$ (an audio signal) has the $|X(j\omega)|$ shown below:



then the magnitude of the Fourier transform of $x(t)\cos(\omega_0 t)$ (with $\omega_0 = 2\pi \times \underbrace{10^6}_{1\text{ MHz}}$ rad/sec) is:



The spectrum is shifted forwards by $+\omega_0$ and backwards by $-\omega_0$. ■



5.3.7 Multiplication by t^n

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) \text{ (\& if } t^n x(t) \text{ is absolutely integrable)}$$

then

$$\boxed{t^n x(t) \overset{\mathscr{F}}{\longleftrightarrow} j^n \frac{d^n X(j\omega)}{d\omega^n}} \tag{5.26}$$

Proof.

$$\begin{aligned} j^n \frac{d^n X(j\omega)}{d\omega^n} &= j^n \frac{d^n}{d\omega^n}\left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\right] \\ &= j^n \int_{-\infty}^{\infty} x(t) \frac{d^n}{d\omega^n}\left(e^{-j\omega t}\right) dt \\ &= j^n \int_{-\infty}^{\infty} x(t)(-jt)^n e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[t^n x(t)\right] e^{-j\omega t} dt \end{aligned}$$

□

■ **Example 5.7**

$$e^{-\alpha t}u(t),\ \mathrm{Re}\{\alpha\} > 0 \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{\alpha + j\omega}$$

$$\therefore\ te^{-\alpha t}u(t),\ \mathrm{Re}\{\alpha\} > 0 \overset{\mathscr{F}}{\longleftrightarrow} j\frac{d}{d\omega}\frac{1}{\alpha + j\omega} = \frac{1}{(\alpha + j\omega)^2}$$

■

5.3.8 Differentiation

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then

$$\boxed{\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega X(j\omega)} \tag{5.27}$$

Proof. We start out by noting that $\lim_{t\to\pm\infty}|x(t)| = 0$ since we assume that $x(t)$ is absolutely integrable. Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}dt &= \underbrace{x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega)e^{-j\omega t}dt}_{\substack{\uparrow \\ \text{integrate by parts}}} \\
&= x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= j\omega \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= j\omega X(j\omega)
\end{aligned}$$

☐

Thus differentiation accentuates the high frequency components in the signal.

R The repeated use of this property yields

$$\boxed{\frac{d^n x(t)}{dt^n} \overset{\mathscr{F}}{\longleftrightarrow} (j\omega)^n X(j\omega)} \tag{5.28}$$

5.3.9 Time-Domain Convolution

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then

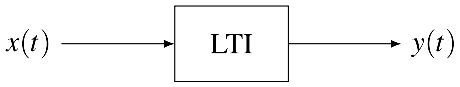
$$\boxed{x_1(t) * x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(j\omega)X_2(j\omega)} \tag{5.29}$$

Proof.

$$\begin{aligned}
\mathscr{F}\{x_1(t) * x_2(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau\right] e^{-j\omega t}dt \\
\text{(interchange the order of integration)} \quad &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(t-\tau)e^{-j\omega t}dt\right] \cdot x_2(\tau)d\tau \\
\text{(make the substitution } \eta = t-\tau) \quad &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\eta)e^{-j\omega\eta}d\eta\right] x_2(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} x_1(\eta)e^{-j\omega\eta}d\eta \cdot \int_{-\infty}^{\infty} x_2(\tau)e^{-j\omega\tau}d\tau \\
&= X_1(j\omega)X_2(j\omega)
\end{aligned}$$

☐

We can apply this result to an LTI system with input $x(t)$, output $y(t)$, impulse response $h(t)$, and frequency response function $H(j\omega)$.



$$h(t) \xleftrightarrow{\mathscr{F}} H(j\omega)$$

Since $y(t) = x(t) * h(t)$, we get

$$\boxed{Y(j\omega) = X(j\omega)H(j\omega)} \tag{5.30}$$

This result is as to be expected! Recall that $x(t)$ is composed of frequency components $X(j\omega)e^{j\omega t}$, and when each of these components passes through an LTI system it comes out as $[H(j\omega)X(j\omega)]\,e^{j\omega t}$. The output $y(t)$ is also composed of frequency components $Y(j\omega)e^{j\omega t}$. Thus,

$$\begin{aligned} Y(j\omega)e^{j\omega t} &= [H(j\omega)X(j\omega)]\,e^{j\omega t} \\ \therefore\ Y(j\omega) &= H(j\omega)X(j\omega) \end{aligned}$$

5.3.10 Frequency-Domain Convolution

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)$$

and

$$y(t) \xleftrightarrow{\mathscr{F}} Y(j\omega)$$

then

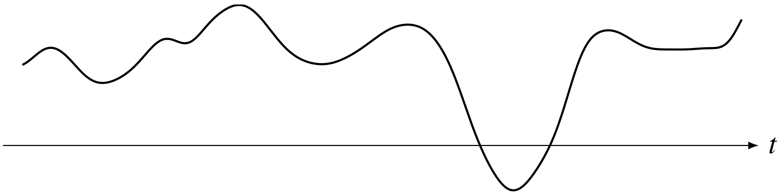
$$\boxed{x(t)y(t) \xleftrightarrow{\mathscr{F}} \frac{1}{2\pi}X(j\omega) * Y(j\omega)} \tag{5.31}$$

Proof.

$$\begin{aligned} \mathscr{F}\{x(t)y(t)\} &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\eta)e^{j\eta t}d\eta\right]e^{-j\omega t}dt \\ \text{(interchange the order of integration)} \quad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\eta)\left[\int_{-\infty}^{\infty} x(t)e^{-j(\omega-\eta)t}dt\right]d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\eta)X(j(\omega-\eta))d\eta \\ &= \frac{1}{2\pi}X(j\omega) * Y(j\omega) \end{aligned}$$

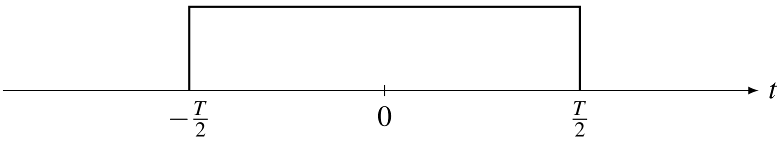
□

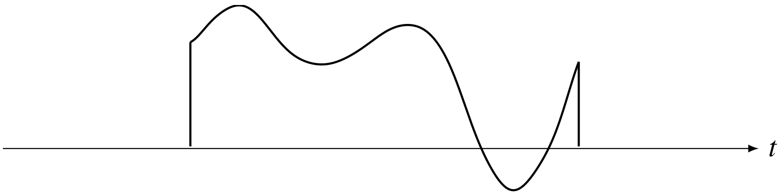
■ Example 5.8 — Effect of Windowing. Consider a signal $x(t)$:



Suppose we only observe it over a window of time $\left(-\frac{T}{2}, \frac{T}{2}\right)$.

$$y(t) = x(t) \cdot \underbrace{\operatorname{rect}\left(\frac{t}{\tau}\right)}_{w(t)}$$





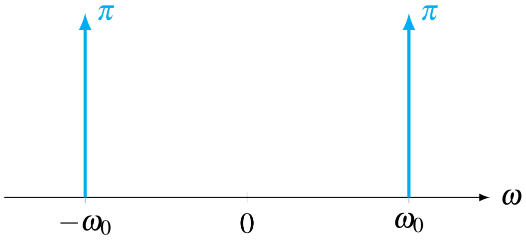
What is the effect of "windowing" in the frequency domain?

$$\begin{aligned} Y(jf) &= \mathscr{F}\{x(t) \cdot w(t)\} \\ &= \frac{1}{2\pi} X(jf) * W(jf) \end{aligned}$$

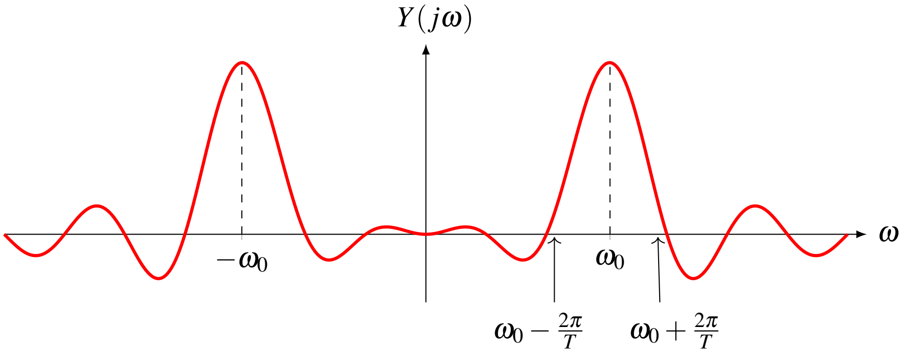
In general, convolution tends to "spread" the signal. ■

■ Example 5.9

$$\begin{aligned} x(t) = \cos(\omega_0 t) &\overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \\ w(t) = \operatorname{rect}\left(\frac{t}{T}\right) &\overset{\mathscr{F}}{\longleftrightarrow} W(j\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) \end{aligned}$$



$$\begin{aligned} Y(j\omega) = X(j\omega) * W(j\omega) &= [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] * T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) \\ &= \pi T \operatorname{sinc}\left[\frac{(\omega - \omega_0) T}{2\pi}\right] + \pi T \operatorname{sinc}\left[\frac{(\omega + \omega_0) T}{2\pi}\right] \end{aligned}$$



Do GNURADIO DEMO: "cos-is-not-delta." ■

5.3.11 Parseval's Theorem for Fourier Transform

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$$

then

$$\boxed{E \equiv \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \tag{5.32}$$

where E is the energy of the signal.

Proof.

$$\begin{aligned}
E &\equiv \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
&= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right]^* dt \\
\text{(interchange the order of integration)} \quad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right] d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}$$

□

In Chapter 4 we derived an analogous relationship (Parseval's Theorem) for Fourier Series. Recall that this relationship states that the total average power of a periodic signal is equal to the sum of the average powers of each of its Fourier series frequency components, i.e.

$$\begin{aligned}
P_{av} \equiv \frac{1}{T}\int_{<T>} |x(t)|^2 dt &= \sum_{n=-\infty}^{\infty} |c_n|^2 \longleftarrow \boxed{\text{general case}} \\
&= c_0^2 + \sum_{n=1}^{\infty} \underbrace{2|c_n|^2}_{\substack{\text{average power in} \\ n^{\text{th}} \text{ harmonic} \\ 2|c_n|\cos\left(n\cdot\frac{2\pi}{T}t + \angle c_n\right)}} \longleftarrow \boxed{\text{if } x(t) \text{ is real-valued}}
\end{aligned}$$

where

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \\
&= c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos\left(\frac{2\pi}{T}nt + \angle c_n\right) \quad \text{(when } x(t) \text{ is real valued)}
\end{aligned}
$$

Definition 5.2 — Energy Spectral Density. Since

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

we can call $\frac{1}{2\pi}|X(j\omega)|^2$ the ***energy spectral density*** of the signal $x(t)$ (in the same way that mass = $\int_{\text{Volume}}$ density).

↑
distribution of energy
across different
frequencies

$x(t)$	$\xleftrightarrow{\text{F.T.}}$	$X(j\omega)$	Eq.
$\alpha x_1(t) + \beta x_2(t)$	$\xleftrightarrow{\mathscr{F}}$	$\alpha X_1(j\omega) + \beta X_2(j\omega)$	(5.18)
If $x(t)$ is real-valued, then		$X(-j\omega) = X^*(j\omega)$	(5.19)
$X(jt)$	$\xleftrightarrow{\mathscr{F}}$	$2\pi x(-\omega)$	(5.20)
$x(t-t_0)$	$\xleftrightarrow{\mathscr{F}}$	$e^{-j\omega t_0}X(j\omega)$	(5.21)
$x(\alpha t)$	$\xleftrightarrow{\mathscr{F}}$	$\frac{1}{\|\alpha\|}X\left(\frac{j\omega}{\alpha}\right)$	(5.22)
$x(t)e^{j\omega_0 t}$	$\xleftrightarrow{\mathscr{F}}$	$X\left(j(\omega-\omega_0)\right)$	(5.23)
$x(t)\cos\omega_0 t$	$\xleftrightarrow{\mathscr{F}}$	$\frac{1}{2}\left[X\left(j(\omega-\omega_0)\right) + X\left(j(\omega+\omega_0)\right)\right]$	(5.24)
$x(t)\sin\omega_0 t$	$\xleftrightarrow{\mathscr{F}}$	$\frac{1}{2j}\left[X\left(j(\omega-\omega_0)\right) - X\left(j(\omega+\omega_0)\right)\right]$	(5.25)

Table 5.2: Table of Fourier Transform Properties

■ **Example 5.10**

$$x(t) = e^{-2t}u(t) \xleftrightarrow{\mathscr{F}} X(j\omega) = \frac{1}{2+j\omega}$$

$$\text{one-sided ESD} = \frac{1}{\pi}|X(j\omega)|^2 = \frac{1}{\pi}\frac{1}{4+\omega^2}$$

$$E = \int_0^{+\infty} \frac{1}{\pi}\frac{1}{4+\omega^2} d\omega$$

Using the trigonometric subsitution $\omega = 2\tan\theta$ and evaluating the integral, we obtain

$$E = \int_0^{\frac{\pi}{2}} \frac{1}{2\pi} d\theta = \frac{1}{4}$$

What is the energy in $\omega \in [0,2]$ (rad/sec)?

$$E_{[0,2]} = \int_0^2 \frac{1}{\pi}\frac{1}{4+\omega^2} d\omega = \cdots = \int_0^{\frac{\pi}{4}} \frac{1}{2\pi} d\theta = \frac{1}{8}$$

so half of the energy is contributed by frequencies $\omega \in [0,2]$. ■

6. Filters and Filtering

The following paragraphs have been taken from Oppenheim and Wilksky's textbook.

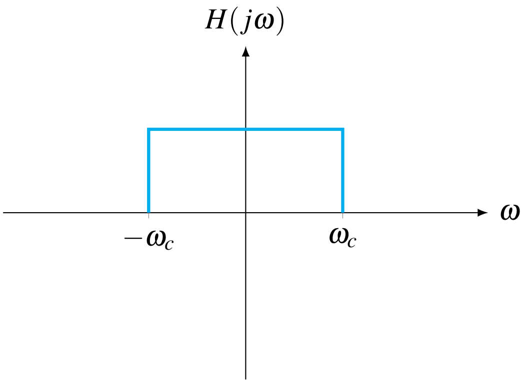
"In a variety of applications, it is of interest to change the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, a process referred to as ***filtering***. For linear-time invariant systems, the spectrum $Y(j\omega) = H(j\omega)X(j\omega)$ (i.e. the Fourier transform) of the output is that of the input multiplied by the frequency response of the system. Consequently, filtering can be conveniently accomplished through the use of such systems with an appropriately chosen frequency response (i.e. transfer function). This represents one of the very important applications of linear time-invariant systems.

One example in which linear time-invariant filtering is encountered is in audio systems. In such systems, a filter is typically included to permit the listener to modify the relative amounts of low-frequency energy (bass) and high-frequency energy (treble). The filter corresponds to a linear time-invariant system whose frequency response is changed by manipulating the tone controls. Also, in high-fidelity audio systems, a filter is often included in the pre-amplifier to compensate for the frequency-response characteristics of the speakers. This filter is called an ***equalizer***.

6.1 Ideal Frequency Selective Filters

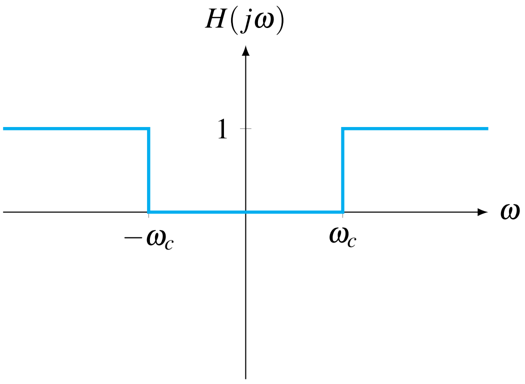
An ***ideal frequency-selective filter*** is one that passes complex exponentials of certain frequencies without change and completely stops complex exponentials of all other frequencies. The range of frequencies that are passed without change are said to lie in the **passband** of the filter, and the range of frequencies with are not passed constitute the **stopband**. Frequency-selective filters are classified as **low-pass**, **high-pass**, **band-pass**, or **band-stop** filters, depending on the band of frequencies that are either passed through without change or completely stopped. The different types of ideal frequency selective filters are illustrated in the figure below.

Low-Pass Filter



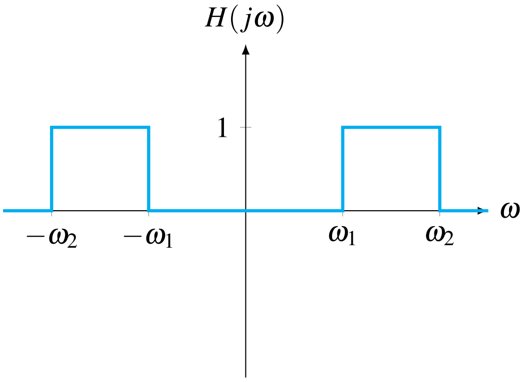
$|\omega| < \omega_c$ passband
$|\omega| > \omega_c$ stopband
$\omega_c =$ "cut-off" frequency

High-Pass Filter



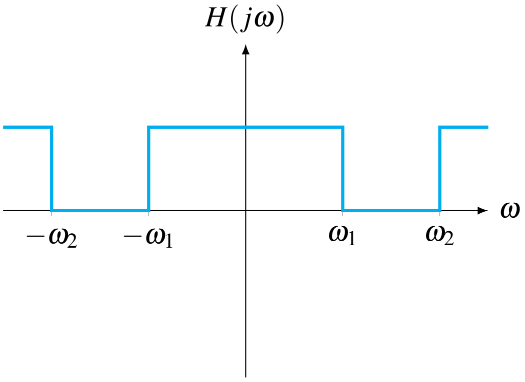
$|\omega| > \omega_c$ passband
$|\omega| < \omega_c$ stopband

Bandpass Filter



$$\begin{array}{l} \omega_1 < |\omega| < \omega_2 \text{ passband} \\ \left.\begin{array}{l} |\omega| < \omega_1 \\ |\omega| > \omega_2 \end{array}\right\} \text{stopband} \end{array}$$

Bandstop Filter

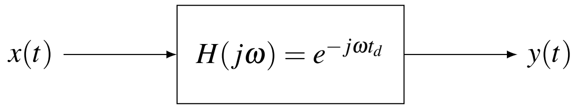


$$\begin{array}{l} \left.\begin{array}{l} |\omega| < \omega_1 \\ |\omega| > \omega_2 \end{array}\right\} \text{passband} \\ \omega_1 < |\omega| < \omega_2 \text{ stopband} \end{array}$$

6.2 Nonideal Frequency Selective Filters

The ideal filters we just considered had abrupt cutoffs. Furthermore, their frequency responses were real-valued and nonnegative, thus $\sum H(j\omega) = 0$. In practice, however, it is not possible to build filters having either of these *characteristics*.

1. First we note that if a filter has a **linear phase**, that is $\angle H(j\omega) = -\omega t_d$, where t_d is a constant, then this phase characteristic will not distort the signal in any way. It will simply **delay** the signal by t_d seconds.



$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_d} X(j\omega) \\ \therefore\ y(t) &= x(t - t_d) \quad \text{(see properties of four)} \end{aligned}$$

Thus when we build frequency-selective filters the goal is to achieve a phase response ($\angle H(j\omega)$) which is nearly linear.

Definition 6.1 — Filter Delay. If a filter has frequency response

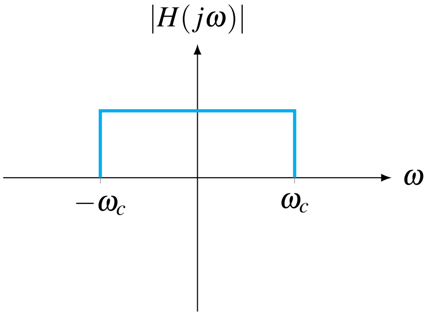
$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

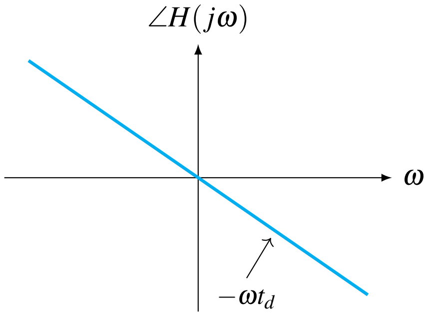
then the ***filter delay*** is defined as

$$D(j\omega) = \frac{d\left[\angle H(j\omega)\right]}{d\omega} \tag{6.1}$$

For a **distortionless** filter, the filter delay is a constant with respect to ω.

2. In practice it is not possible to build filters which have abrupt transitions from the passbands to the stopbands. For example, consider the ideal lowpass filter shown below.





The impulse response, $h(t)$, of this filter is:

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega \\ &= \frac{\omega_c}{\pi} \frac{\sin\left[\omega_c (t - t_d)\right]}{\omega_c (t - t_d)} \\ &= \frac{\omega_c}{\pi} \operatorname{sinc}\left[\frac{\omega_c (t - t_d)}{\pi}\right] \end{aligned}$$

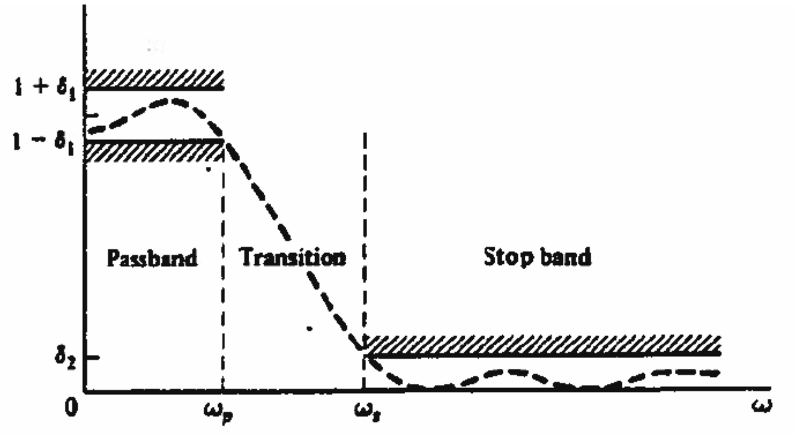
$h(t) \not\equiv 0$ for $t < 0 \Longleftarrow$ ***This filter is not causal and therefore it cannot be built!***

Thus, in order to get a realizable filter we must relax our requirements on $|H(j\omega)|$. The specification for a **realizable** low pass filter, for example, might become

$$1 - \delta_1 \le |H(j\omega)| \le 1 + \delta_1, \quad |\omega| \le \omega_p \tag{6.2}$$

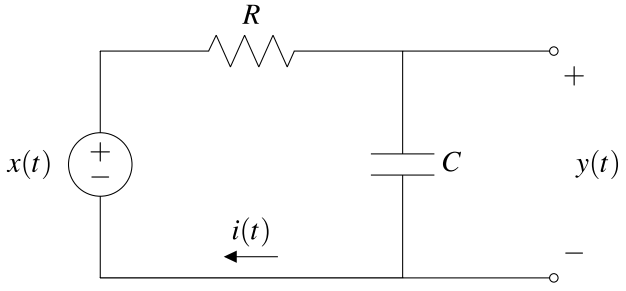
$$|H(j\omega)| \le \delta_2, \quad |\omega| \ge \omega_s \tag{6.3}$$

where ω_p and ω_s are the nominal **pass-band** and **stop-band cut-off frequencies**, respectively. The range of frequencies between ω_p and ω_s are called the ***transition band***.



■ **Example 6.1** A filter is said to be ***passive*** if it has no gain/amplifying elements (i.e. elements that can increase signal power). ***Active filters*** contain gain/amplifying elements, such as transistors. At low frequencies, active filters constructed from op-amps, resistors, and capacitors are quite common. In Lab IV: AM Radio, you will build and characterize an active bandpass filter known as an *IF* (i.e. **intermediate frequency**) *filter*.

Consider the passice RC lowpass filter shown below:



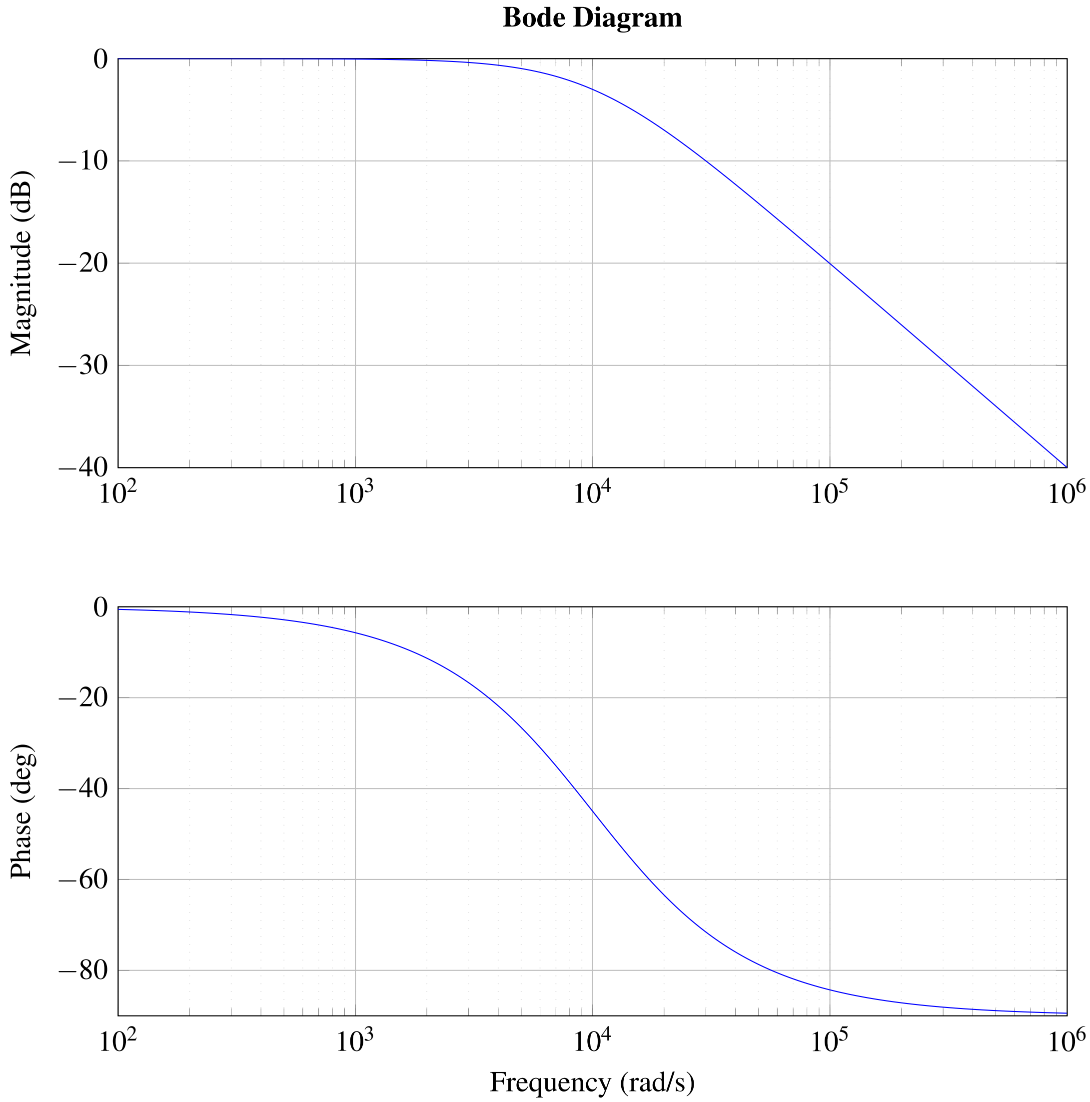
The frequency transfer function, $H(j\omega)$ of this filter is easily derived using standard circuit analysis methods:

$$H(j\omega) = \frac{1/(j\omega C)}{1/(j\omega C)+R} = \frac{1}{1+j\omega RC}$$

The magnitude in dB (i.e. $20\log_{10}|H(j\omega)|$) and the phase response (i.e. $\angle H(j\omega)$) are plotted vs. frequency ω on the next page for the case when $RC = 10^{-4}$ s. Unlike an ideal low-pass filter, this filter does not have a "sharp" frequency cut-off. The **3 dB frequency cut-off** (i.e. ω for which $20\log_{10}|H(j\omega)| = -3$ dB) is at 10^4 rad/s and beyond this frequency, the magnitude response (i.e. $20\log_{10}|H(j\omega)|$) *decreases by approximately* 20 *dB/decade*, where a decade is a multiplicative factor of 10 in frequency.

Plots of $20\log_{10}|H(j\omega)|$ vs. ω and $\angle H(j\omega)$ vs. ω are known as ***Bode plots***.

A log scale is used for the frequency axis to permit $20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ to be plotted over a large frequency range. Furthermore, $20\log_{10}|H(j\omega)|$, rather than $|H(j\omega)|$, is plotted to accomodate a large **dynamic range**. If $|H(j\omega)|$ vs. ω were plotted instead, then it would be difficult to read from the plot small (e.g. 10^{-4}) values of $|H(j\omega)|$.



■

6.3 Matlab `bode` Command

If $H(j\omega)$ can be expressed as (known as a **rational form**)

$$H(j\omega) = \frac{N(s)}{D(s)}, \quad s \equiv j\omega \tag{6.4}$$

where $N(s)$ and $D(s)$ are polynomials in s with real-valued coefficients, then there is a simple Matlab command (`bode`) that will generate the Bode plots of $H(j\omega)$. The procedure is as follows:

1. Write $N(s)$ and $D(s)$ as vectors, N and D, where the vectors consist of the polynomial coefficients in descending order.
2. Use the Matlab command `bode(N,D)`

■ Example 6.2

$$H(j\omega) = \frac{10^{10}}{(10^{10}-\omega^2)+j\sqrt{2}10^5\omega} = \frac{10^{10}}{(j\omega)^2+\sqrt{2}10^5(j\omega)+10^{10}}$$

(2^{nd} order Butterworth Low-pass filter).

$$\therefore\ H(j\omega) = \frac{N(s)}{D(s)}$$

where $N(s) = 10^{10}$ and $D(s) = s^2 + \sqrt{2}10^5 s + 10^{10}$.

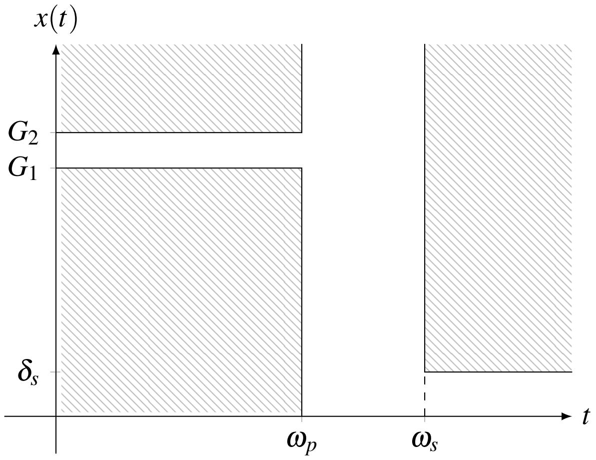
$$\therefore\ \texttt{N} = [\texttt{1.0E10}]\ \&\ \texttt{D} = [\texttt{1 1.414E05 1.0E10}]$$

Use the Matlab command `bode(N,D)`. ■

6.4 Systematic Filter Design

Filter design specifications are given in terms of a "spectral" mask.

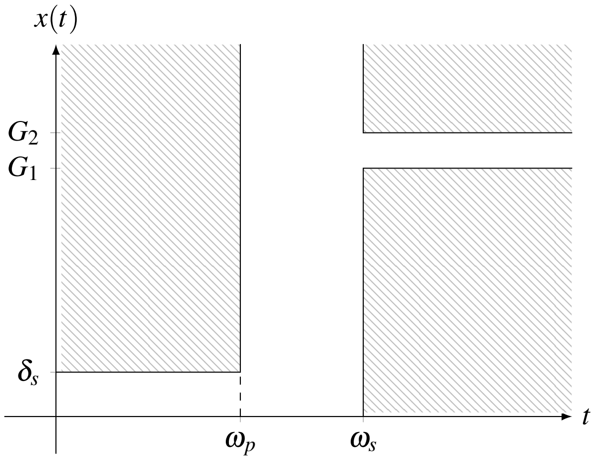
1. LPF



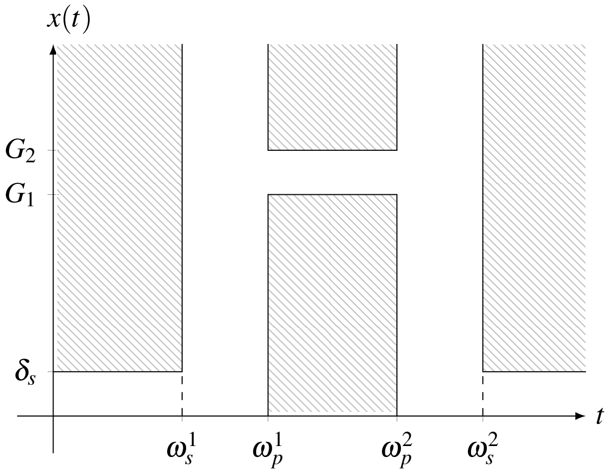
- Any design $H(j\omega)$ with $|H(j\omega)|$ "satisfying" the spectral mask is acceptable.
- The design specs are augmented by
  - as linear as possible $\angle H(j\omega)$ for ω in passband
  - as low cost as possible (usually manifesting itself in the max. degree of the polynomials $N(j\omega)$ and $D(j\omega)$ of $H(j\omega) = \frac{N(j\omega)}{D(j\omega)}$).

Similarly, masks for HPF, BPF, and BSF.

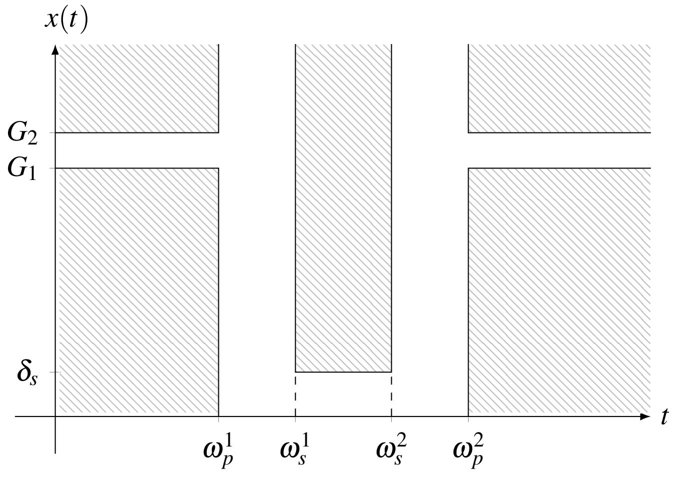
2. HPF



3. BFP



4. BSP



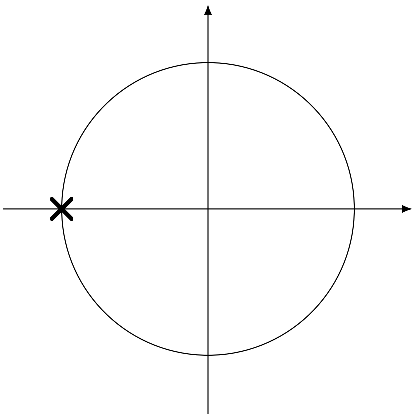
- We will build LPF, HPF, BPF, and BSF starting from a "prototype LPF" and applying to it certain transformations.

- An example of a "prototype LPF" is a Butterworth filter of order N.

$$H(j\omega) = \frac{1}{\prod_{k=1}^{N}(j\omega - s_k)} \tag{6.5}$$

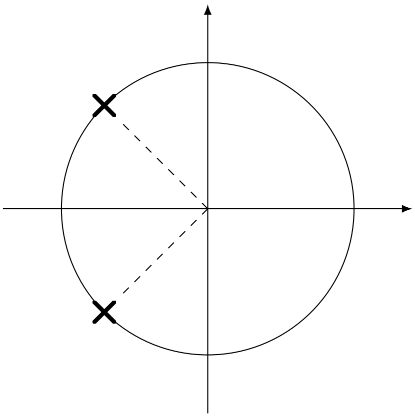
where $s_k = e^{j\frac{(2k+N-1)\pi}{2N}}$, $k = 1, 2, \ldots, N$.

■ **Example 6.3** $N = 1$:



$$H(j\omega) = \frac{1}{j\omega + 1}$$

$N = 2$:



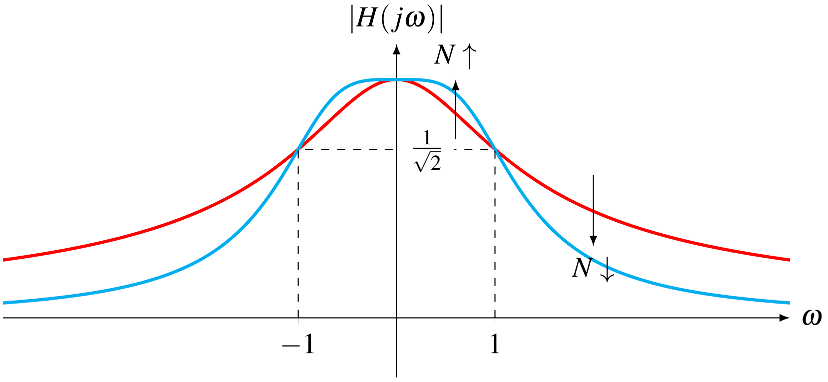
$$\begin{aligned} H(j\omega) &= \frac{1}{\left(j\omega - e^{j3\pi/4}\right)\left(j\omega - e^{j5\pi/4}\right)} \\ &= \frac{1}{(j\omega)^2 + \sqrt{2}(j\omega) + 1} \end{aligned}$$

■

6.4.1 Properties of Butterworth Filters

The Butterworth family of filters was defined earlier through its FRF (6.5). As it turns out (this is not straightforward, but a good exercise) the magnitude of this FRF is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2N}}} \tag{6.6}$$

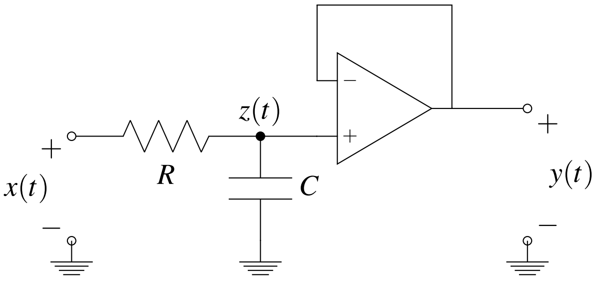


As can be easily checked, $H(j0)=1$, $H(j1)=1/\sqrt{2}$ and $\lim_{\omega\to\pm\infty}|H(j\omega)|=0$. As N increases, we get a flatter FRF in the passband and a sharper transition to the stopband.

How can we implement Butterworth filters with R, C components and opamps?

■ **Example 6.4** $N=1$:

$$H(j\omega)=\frac{1}{1+j\omega}$$

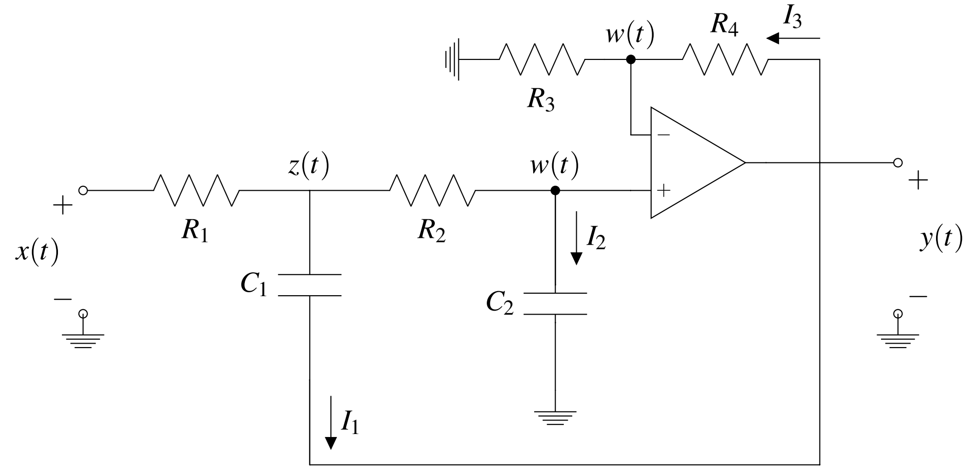


From the golden rules of op-amps,

$$\left.\begin{aligned}&\tilde{Z}=\tilde{Y}\\&\tilde{Z}=\tilde{X}\cdot\frac{1/(Cj\omega)}{R+1/(Cj\omega)}=\tilde{X}\cdot\frac{1}{1+RCj\omega}\end{aligned}\right\}\implies H(j\omega)=\frac{1}{1+\underbrace{RC}_{=1}j\omega}$$

In general, set $\omega_0=\frac{1}{RC}\implies\hat{H}(j\omega)=\frac{1}{1+j\frac{\omega}{\omega_0}}$. Note that the "buffer" attached at the end of the filter is not required if the filter is used as is, but it is useful when concatenating several filters in series, so that one stage does not "load" the previous one. ■

■ **Example 6.5** $N=2$:



$$\tilde{I}_1 = \frac{\tilde{Z}-\tilde{Y}}{1/(C_1 j\omega)},\ \tilde{I}_2 = \frac{\tilde{W}}{1/(C_2 j\omega)},\ I_3 = \frac{\tilde{W}}{R_3}$$

$$\tilde{Y} = \tilde{W} + I_3 R_4 = \tilde{W} + \frac{\tilde{W}}{R_3}R_4 = W\underbrace{\left(1+\frac{R_4}{R_3}\right)}_{A} \tag{6.7}$$

$$\tilde{Z} = \tilde{X} - R_1(I_1+I_2) \implies \tilde{Z} = \tilde{X} - R_1\left[(\tilde{Z}-\tilde{Y})C_1 j\omega + \tilde{W}C_2 j\omega\right] \tag{6.8}$$

$$\tilde{W} = \tilde{Z} - I_2 R_2 \implies \tilde{W} = \tilde{Z} - R_2\tilde{W}C_2 j\omega \tag{6.9}$$

$$(6.9) \implies (1+R_2C_2 j\omega)\tilde{W} = \tilde{Z} \tag{6.10}$$

$$(6.8) \implies \tilde{Z} = \tilde{X} - \tilde{Z}\cdot R_1C_1 j\omega + \tilde{Y}R_1C_1 j\omega - \tilde{W}R_1C_2 j\omega$$

$$\implies (1+R_1C_1 j\omega)\tilde{Z} = \tilde{X} + \tilde{Y}R_1C_1 j\omega - \tilde{W}R_1C_2 j\omega \tag{6.11}$$

$$(6.11),(6.10) \implies (1+R_1C_1 j\omega)(1+R_2C_2 j\omega)\tilde{W} = \tilde{X} + \tilde{Y} + R_1C_1 j\omega - \tilde{W}R_1C + 2j\omega$$

$$\implies \left[(1+R_1C_1 j\omega)(1+R_2C_2 j\omega) + R_1C_2 j\omega\right]\tilde{W} = \tilde{X} + \tilde{Y}\cdot R_1C_1 j\omega$$

$$(6.7) \implies [\ ]\frac{\tilde{Y}}{A} = \tilde{X} + \tilde{Y}\cdot R_1C_1 j\omega$$

$$\implies \left(\frac{[\ ]}{A} - R_1C_1 j\omega\right)\tilde{Y} = \tilde{X}$$

$$\implies H(j\omega) = \frac{1}{\frac{[\ ]}{A} - R_1C_1 j\omega}$$

$$= \frac{A}{(1+R_1C_1 j\omega)(1+R_2C_2 j\omega) + R_1C_2 j\omega - AR_1C_1 j\omega}$$

$$= \frac{A}{1 + (R_1C_1 + R_2C_2 + R_1C_2 - AR_1C_1)j\omega + R_1C_1R_2C_2(j\omega)^2}$$

Specialize $R_1 = R_2 = R$, $C_1 = C_2 = C$:

$$H(j\omega) = \frac{A}{1 + (3-A)RCj\omega + (RC)^2(j\omega)^2}$$

$$\downarrow \text{ set } RC = 1,\ 3 - A = \sqrt{2}$$

$$H(j\omega) = \frac{\overbrace{3-\sqrt{2}}^{=1.59}}{1+\sqrt{2}(j\omega)+(j\omega)^2} \approx 2^{\text{nd}} \text{ order Butterworth filter}$$

In general, $\omega_0 = \frac{1}{RC}$, $3 - A = \sqrt{2}$:

$$\hat{H}(j\omega) = \frac{1.56}{1 + \sqrt{2}\left(j\frac{\omega}{\omega_0}\right) + \left(j\frac{\omega}{\omega_0}\right)^2}$$

■

■ **Example 6.6** For an arbitrary N one can design a prototype Butterworth filter by serially concatenating several filters together (each of order 2) followed by possibly a filter if order 1. Specifically, consider first the case when N is an even number. One can rewrite the FRF $H(j\omega) = \frac{1}{\prod_{k=1}^{N}(j\omega - s_k)}$, where $s_k = e^{j\frac{(2k+N-1)\pi}{2N}}$, $k = 1, 2, \ldots, N$, as

$$H(j\omega) = \prod_{n=1}^{N/2} \frac{1}{(j\omega - s_n)(j\omega - s_n^*)} \tag{6.12}$$

$$= \prod_{n=1}^{N/2} \frac{1}{|s_n|^2 - (j\omega)2\Re[s_n] + (j\omega)^2} \tag{6.13}$$

$$= \prod_{n=1}^{N/2} \frac{1}{1 - (j\omega)2\cos(\frac{(2n+N-1)\pi}{2N}) + (j\omega)^2}. \tag{6.14}$$

Each of the above $N/2$ factors represents an 2nd order filter that can be realized similar to the one in Example 6.5 with $RC = 1$ and $3 - A = 2\cos(\frac{(2n+N-1)\pi}{2N})$.

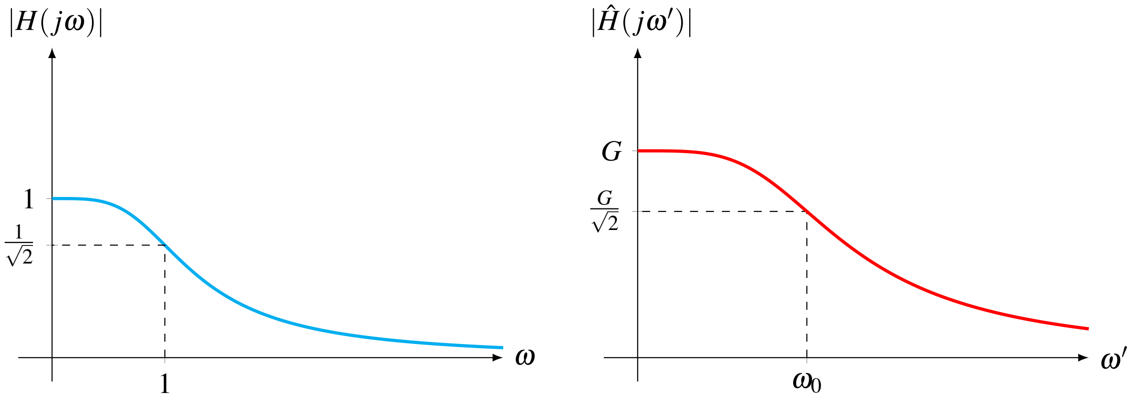
Similarly, for the case where N is odd, the overall FRF can be written as above together with a factor $\frac{1}{1+j\omega}$ which corresponds to the 1st order filter described earlier. ■

6.4.2 Design process through transformations of the prototype filter

Design of LPF

We want to design of a LPF that satisfies the spectral mask given by (ω_p, ω_s, G_1, G_2, δ_s) from a prototype LPF.

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2N}}} \xrightarrow[\omega \to \frac{\omega'}{\omega_0}]{j\omega \to j\frac{\omega'}{\omega_0}} |\hat{H}(j\omega')| = \frac{G}{\sqrt{1+\left(\frac{\omega'}{\omega_0}\right)^{2N}}} \tag{6.15}$$



To complete the design, we need to find the parameters G, ω_0, N that satisfy the following inequalities (mask)

$$\left.\begin{array}{r} G_1 \le G \le G_2 \\ \left|\hat{H}(j\omega_p)\right| \ge G_1 \\ \left|\hat{H}(j\omega_s)\right| \le \delta_s \end{array}\right\} \text{ solve for } (G, \omega_0, \underbrace{N}_{\text{minimal integer}}).$$

Since solving a set of inequalities is difficult, we can consider a set of equations that will exactly match the mask, as follows:

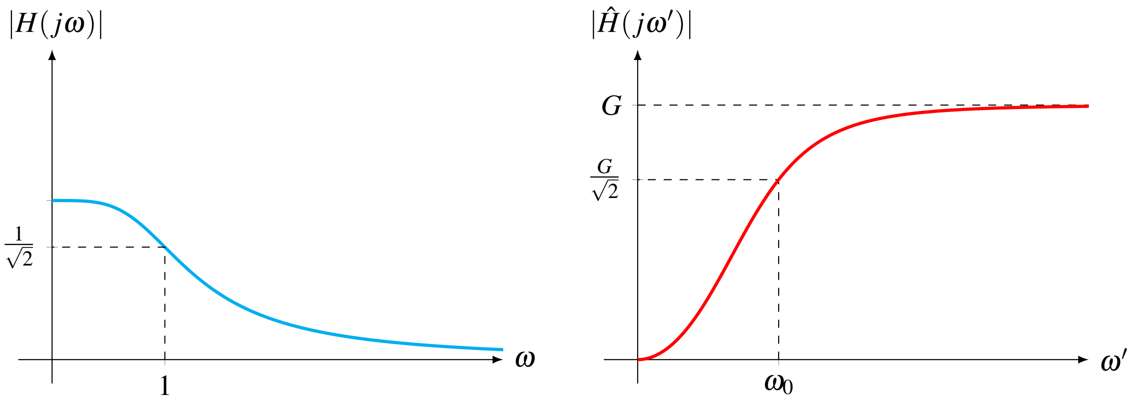
$$\left.\begin{array}{r} G = G_2 \\ \left|\hat{H}(j\omega_p)\right| = G_1 \\ \left|\hat{H}(j\omega_s)\right| = \delta_s \end{array}\right\} \text{ solve for } (G, \omega_0, \underbrace{N}_{\text{minimal integer}})$$

Interpretation of $j\omega \to j\frac{\omega'}{\omega_0}$: every resistor/capacitor is scaled appropriately.

Design of HPF

$$H(j\omega) = \frac{1}{\sqrt{1+\omega^{2N}}} \xrightarrow[\omega \to -\omega_0/\omega']{j\omega \to \frac{1}{j\omega'/\omega_0}} \left|\hat{H}(j\omega')\right| = \frac{G}{\sqrt{1+\frac{\omega_0^{2N}}{\omega'^{2N}}}} \tag{6.16}$$

$$= G\sqrt{\frac{\omega'^{2N}}{\omega'^{2N}+\omega_0^{2N}}} \tag{6.17}$$



To complete the design, we need to find the parameters G, ω_0, N that satisfy the inequalities describing the given mask.

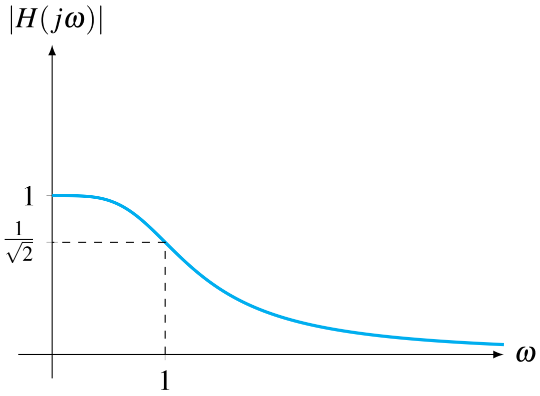
Interpretation of $j\omega \longrightarrow \frac{1}{j\omega'/\omega_0}$:

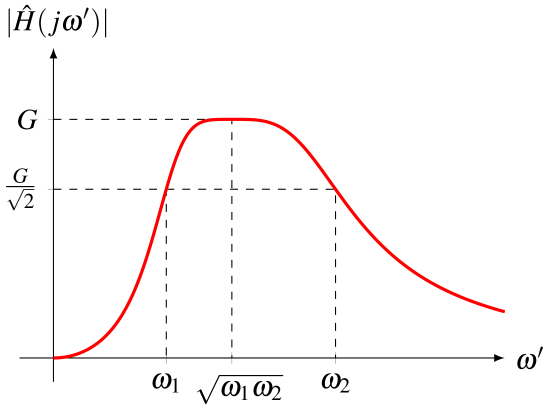
- Every capacitor (inductor) in the prototype filter is substituted by an appropriately scaled inductor (capacitor) in the new filter.
- Alternatively every R, C in the RC stage of the filter (not in the inverting amplifier) is substituted with a $\hat{C}$, $\hat{R}$.

Design of BPF

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2N}}} \xrightarrow[j\omega \to j\frac{\omega'}{\omega_2-\omega_1} + \frac{\frac{\omega_1\omega_2}{(\omega_2-\omega_1)^2}}{j\frac{\omega'}{\omega_2-\omega_1}}]{} \left|\hat{H}(j\omega')\right| = \frac{G}{\sqrt{1+\left(\frac{\omega'^2-\omega_1\omega_2}{\omega'(\omega_2-\omega_1)}\right)^{2N}}} \tag{6.18}$$

$$\omega \longrightarrow \frac{\omega'}{\omega_2-\omega_1} - \frac{\omega_1\omega_2}{\omega'(\omega_2-\omega_1)} = \frac{\omega'^2-\omega_1\omega_2}{\omega'(\omega_2-\omega_1)} \tag{6.19}$$





$$\omega = 0 \longrightarrow \omega' = \sqrt{\omega_1\omega_2}$$
$$\omega = \pm 1 \longrightarrow \omega' = \pm\omega_1,\ \pm\omega_2$$

To complete the design, we need to find the parameters G, ω_1, ω_2, N that satisfy the inequalities describing the given mask.

Interpretation of $j\omega \rightarrow j\frac{\omega'}{\omega_2-\omega_1} + \frac{\frac{\omega_1\omega_2}{(\omega_2-\omega_1)^2}}{j\frac{\omega'}{\omega_2-\omega_1}}$:

- Every $Cj\omega$ in the prototype is substituted by a $\hat{C}j\omega + \frac{1}{\hat{L}j\omega}$

Design of BSF

Consider the two consecutive transformations

$$\left.\begin{aligned} j\omega \rightarrow \frac{1}{j\omega'} \quad \left(\omega \rightarrow -\frac{1}{\omega'}\right) &: \text{makes the prototype LPF a prototype HPF} \\ j\omega' \rightarrow j\frac{\omega''}{\omega_2-\omega_1} + \frac{\frac{\omega_1\omega_2}{(\omega_2-\omega_1)^2}}{j\frac{\omega''}{\omega_2-\omega_1}} &: \text{makes the prototype HPF a BSF} \end{aligned}\right\} \Longrightarrow$$

overall,

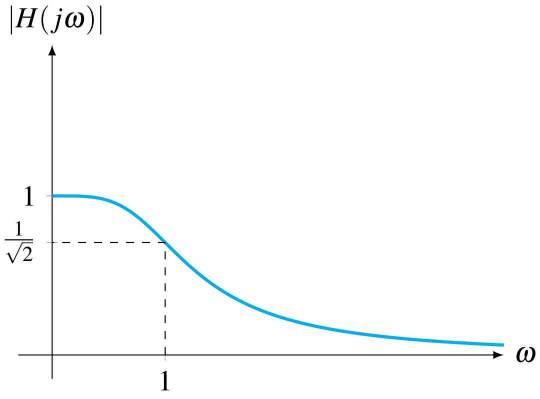
$$j\omega \longrightarrow \left(j\frac{\omega''}{\omega_2-\omega_1} + \frac{\frac{\omega_1\omega_2}{(\omega_2-\omega_1)^2}}{j\frac{\omega''}{\omega_2-\omega_1}}\right)^{-1} \tag{6.20}$$

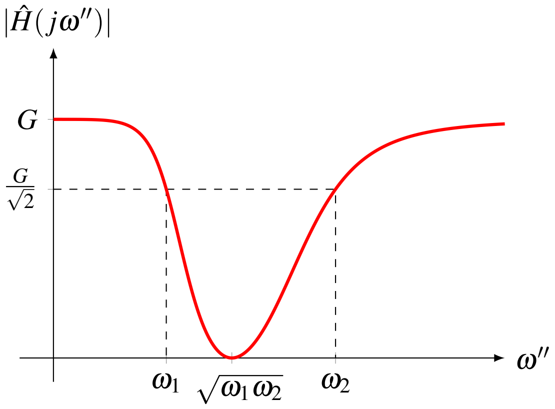
or

$$\omega \longrightarrow \frac{\omega''(\omega_2-\omega_1)}{\omega_1\omega_2 - \omega''^2} \tag{6.21}$$

so

$$\left|\hat{H}(j\omega'')\right| = \frac{G}{\sqrt{1+\left(\frac{\omega''(\omega_2-\omega_1)}{\omega_1\omega_2-\omega''^2}\right)^{2N}}} \tag{6.22}$$

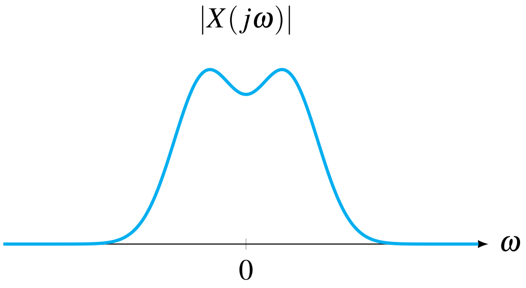




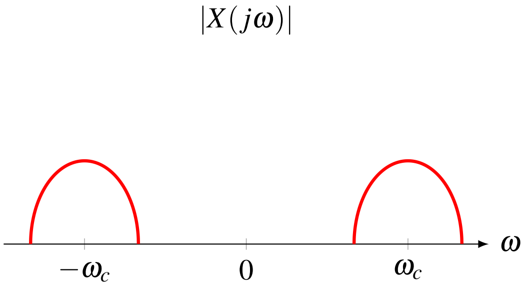
To complete the design, we need to find the parameters G, ω_1, ω_2, N that satisfy the inequalities describing the given mask.

Overall interpretation of the transformation is that every capacitor is substituted by a series concatenation of a capacitor and an inductor.

6.5 Measures of Bandwidth



Baseband



Bandpass

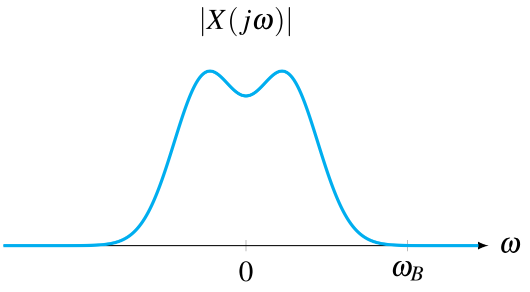
Definition 6.2 — Baseband Signals. Signals whose Fourier Transforms are concentrated around 0 (rad/sec) are called ***baseband signals***.

Definition 6.3 — Bandpass Signals. Signals whose Fourier Transforms are concentrated around some frequency ω_c are called ***bandpass signals***.

Q: How do we measure the bandwidth (BW) of a baseband or bandpass signal?

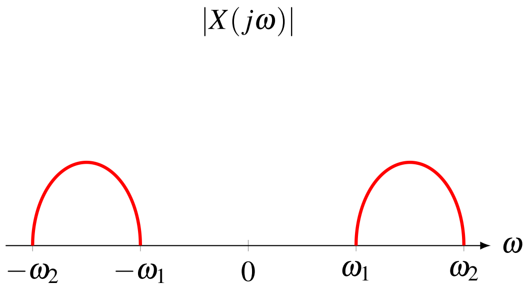
A: There are several bandwidth measures (we will examine 4 of them). For simplicity in exposition, we will assume that $x(t)$ is real so $|X(j\omega)|$ is even.

6.5.1 Absolute Bandwidth



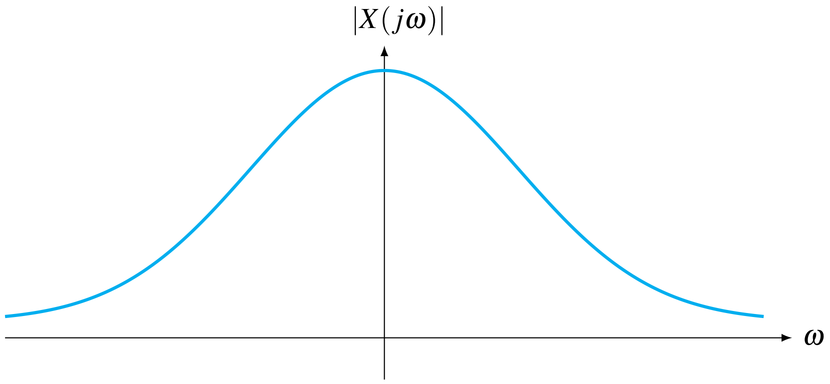
$$BW_{abs} = \omega_B$$

Highest non-zero frequency component of $|X(j\omega)|$.

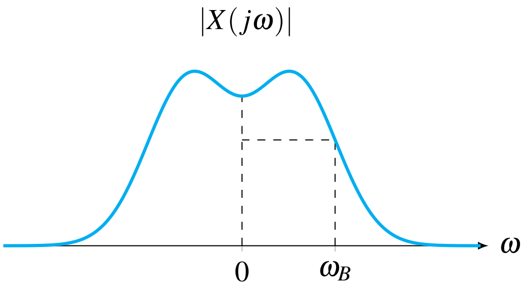


$$BW_{abs} = \omega_2 - \omega_1$$

If $|X(j\omega)|$ extends to ∞ then $BW = \infty$, e.g.



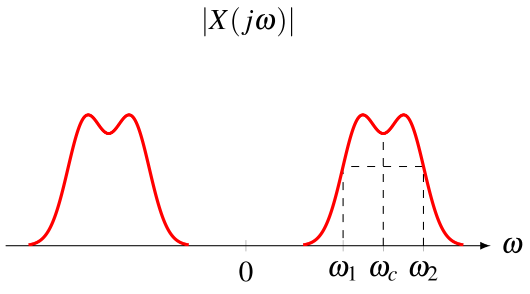
6.5.2 3 dB or Half-Power Bandwidth



$$BW_{3dB} = \omega_B$$

$$\Updownarrow$$

$$|X(j\omega_B)| = \frac{1}{\sqrt{2}}|X(j0)|$$



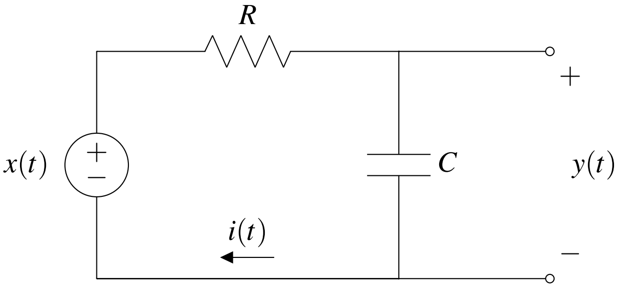
$$BW_{3dB} = \omega_2 - \omega_1$$

ω_1: $|X(j\omega_1)| = \frac{1}{\sqrt{2}}|X(j\omega_c)|$

ω_2: $|X(j\omega_2)| = \frac{1}{\sqrt{2}}|X(j\omega_c)|$

$\omega_1 < \omega_c < \omega_2$

■ **Example 6.7** RC Filter:

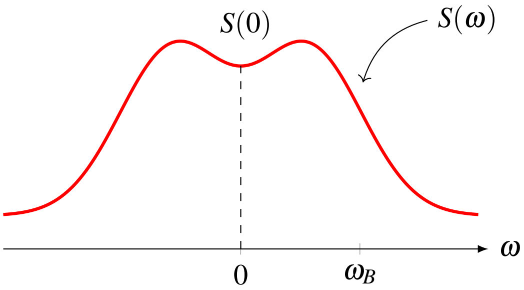


$$
\begin{aligned}
H(j\omega) &= \frac{1}{1+j\omega RC} \\
H(j0) &= 1 \\
|H(j\omega)| &= \frac{1}{\sqrt{1+\omega^2(RC)^2}} \\
|H(j\omega_B)| &= \frac{1}{\sqrt{2}}|H(j0)| \iff \frac{1}{1+\omega_B^2(RC)^2} = \frac{1}{2}\cdot 1 \iff \boxed{\omega_B = \frac{1}{RC}}
\end{aligned}
$$

■

6.5.3 Noise Equivalent Bandwidth

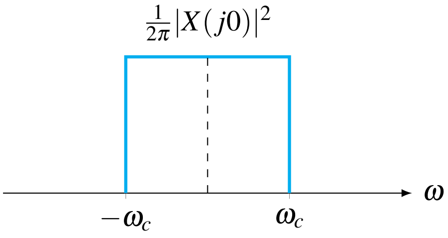
Consider the energy spectral density $S(\omega) = \frac{1}{2\pi}|X(j\omega)|^2$ of a baseband signal.



From Parseval's theorem,

$$
E = \int_{-\infty}^{\infty} S(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega
$$

Consider an "equivalent" e.s.d $\overline{S}(\omega) = \begin{cases} S(0) & |\omega| < \omega_B \\ 0 & \text{else} \end{cases}$



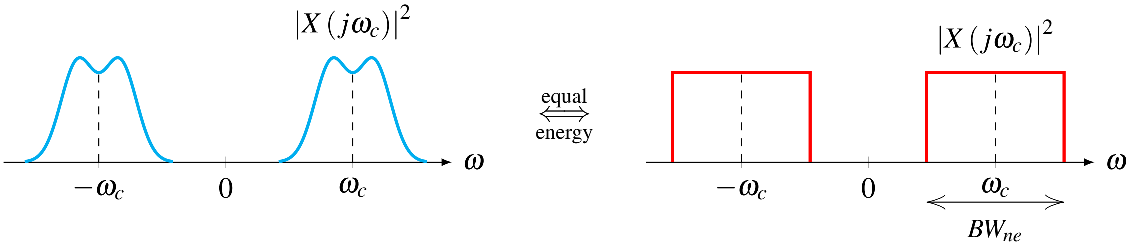
The energy of this signal is $E' = 2\omega_B \frac{1}{2\pi}|X(jd)|^2$. The **noise-equivalent bandwidth** is defined as

$$BW_{ne} = \omega_B \iff E = E' \iff \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega = \frac{1}{\cancel{2\pi}}2\omega_B|X(j\omega)|^2 \tag{6.23}$$

$$\iff \omega_B = \frac{\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega}{2|X(j0)|^2} \tag{6.24}$$

$$\iff \omega_B = \frac{\int_{0}^{+\infty}|X(j\omega)|^2 d\omega}{|X(j0)|^2} \tag{6.25}$$

Similarly for bandpass signals,



$$\int_0^{+\infty}|X(j\omega)|^2 d\omega = BW_{ne}\cdot|X(j\omega_c)|^2 \iff$$

$$BW_{ne} = \frac{\int_0^{+\infty}|X(j\omega)|^2 d\omega}{|X(j\omega_c)|^2} \tag{6.26}$$

6.5.4 The Root-Mean-Squared (RMS) Bandwidth

This is very similar to the idea of "standard deviation" in statistics/probability.

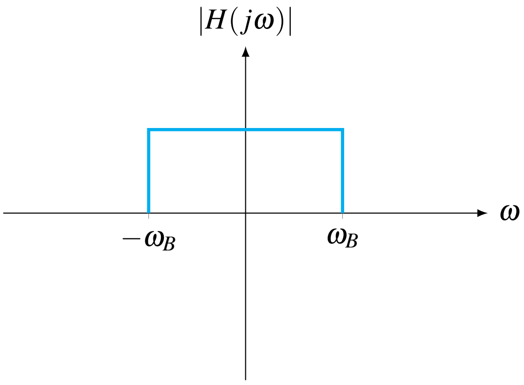
$$BW_{rms} = \sqrt{\frac{\int_0^{+\infty}\omega^2|X(j\omega)|^2 d\omega}{\int_0^{+\infty}|X(j\omega)|^2 d\omega}} \text{ (for baseband)} \tag{6.27}$$

$$BW_{rms} = \sqrt{\frac{\int_0^{+\infty}(\omega-\omega_c)^2|X(j\omega)|^2 d\omega}{\int_0^{+\infty}|X(j\omega)|^2 d\omega}} \text{ (for passband)} \tag{6.28}$$

Here ω_c can be thought of as the "center" of gravity of $|X(j\omega)|^2$ (on the positive side):

$$\omega_c = \frac{\int_0^{+\infty}\omega|X(j\omega)|^2 d\omega}{\int_0^{+\infty}|X(j\omega)|^2 d\omega}$$

■ **Example 6.8** Ideal lowpass filter:



$$BW_{RMS} = \sqrt{\frac{\int_0^{\omega_B} \omega^2 A^2 d\omega}{\int_0^{\omega_B} A^2 d\omega}} = \sqrt{\frac{\cancel{A^2}\, \frac{\omega^3}{3}\Big|_0^{\omega_B}}{\cancel{A^2}\, \omega\big|_0^{\omega_B}}} = \sqrt{\frac{\omega_B^2}{3}} = \frac{\omega_B}{\sqrt{3}}$$

■

In the same way that we defined measures of bandwidth, i.e. how broad $|X(j\omega)|$ is, we can define measures of time duration, i.e. how broad $x(t)$ is.

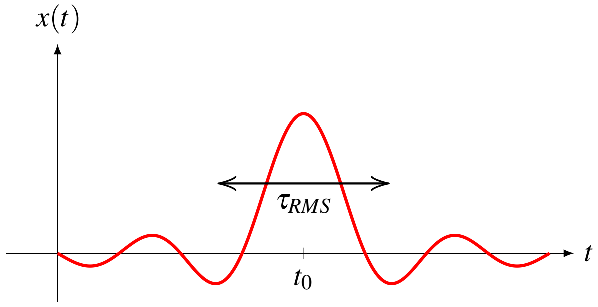
■ **Example 6.9**

Definition 6.4 — RMS Time Duration. The **root-mean-squared (RMS) time duration** of a signal is defined as

$$\tau_{RMS} = \sqrt{\frac{\int_{-\infty}^{\infty} (t-t_0)^2\, |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}} \tag{6.29}$$

where t_0 is an estimate of where the signal has most of its energy and is given by:

$$t_0 = \frac{\int_{-\infty}^{\infty} t \cdot |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \tag{6.30}$$

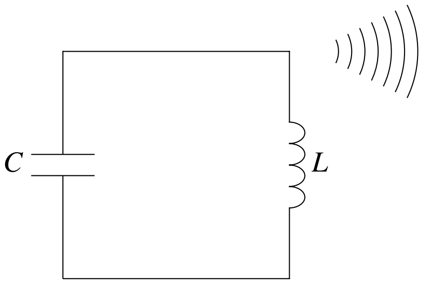


■

7. Amplitude Modulation

7.1 Modulation: What and Why

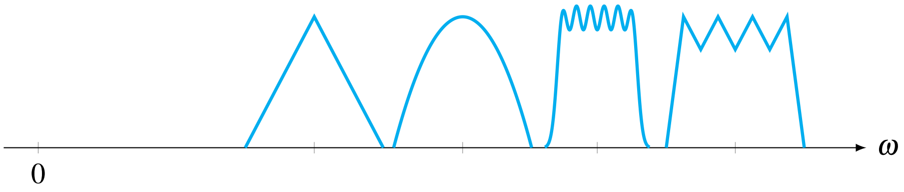
- The process of "***modulation***" involves three signals:
  - $x(t)$, a slowly-varying **information** signal (e.g. audio)
  - $c(t)$, a high-frequency **carrier** signal (e.g. $\cos(\omega_c t)$)
  - $y(t)$, a **modulated** signal which depends on $x(t)$ and $c(t)$
- Modulation is the process by which some attribute of the farrier signal (e.g. amplitude, frequency, etc.) is modified in accordance with the information signal.
- Why modulate?
  Three reasons:
  1. For a physical circuit to "act" as an antenna, its dimension has to be of the same order of the wavelength of its oscillation.



$$c = \lambda f \implies \lambda = \frac{c}{f}$$

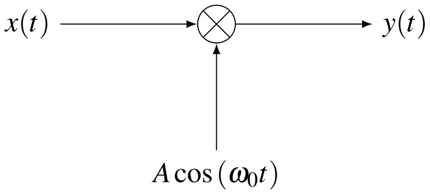
e.g. audio signal, $f = 5$ kHz $\implies \lambda = \frac{3\times10^8}{5\times10^3} =$ **60 km**. But if the signal is modulated with a high-frequency carrier, $f = 1$ MHz and $\lambda = \frac{3\times10^8}{1\times10^6} =$ **300 m**.

  2. The propagation characteristics of the medium of interest may be more favorable for one frequency vs. another. For example, transmission over the atmosphere is very lossy over certain frequency ranges.
  3. Multiple access: We want to have multiple signals transmitted at the same time over a band of frequencies (e.g. AM radio):

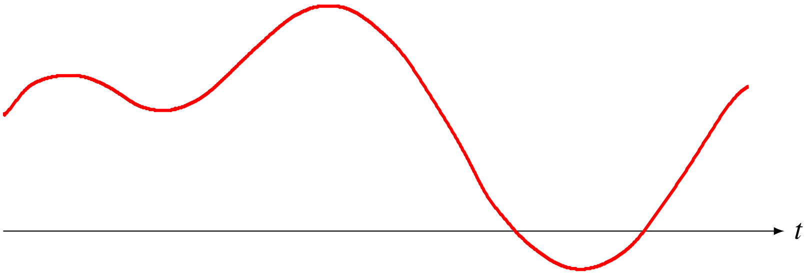


The receiver can now filter out the other signals and process the desired one.

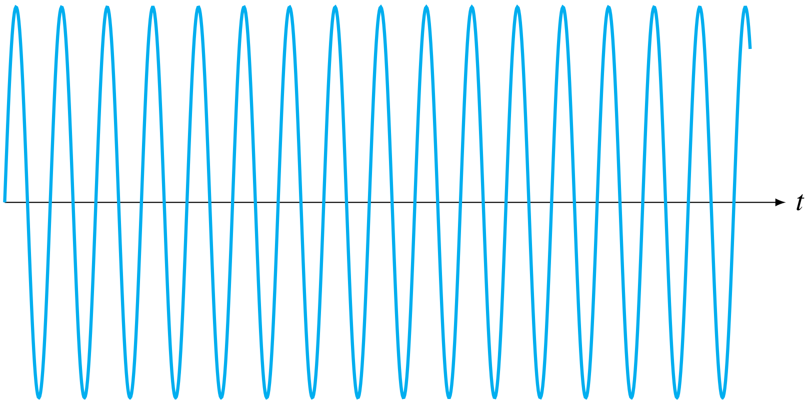
7.2 Suppressed-Carrier Amplitude Modulation (SC-AM)



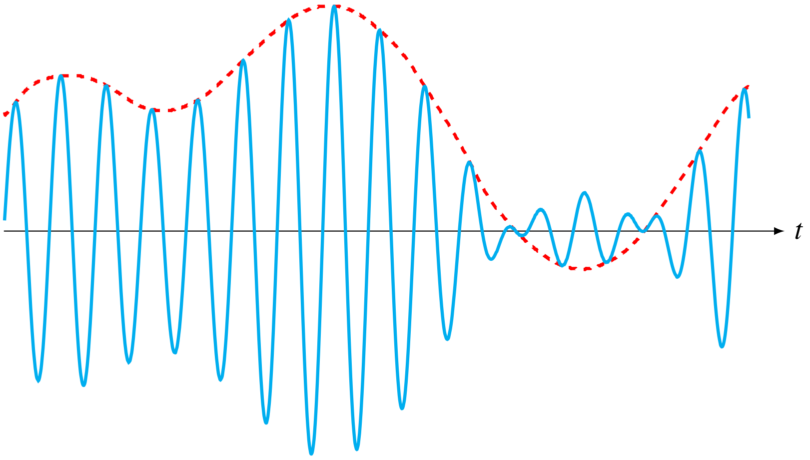
$x(t)$



$A\cos(\omega_c t)$



$y(t) = Ax(t)\cos(\omega_c t)$



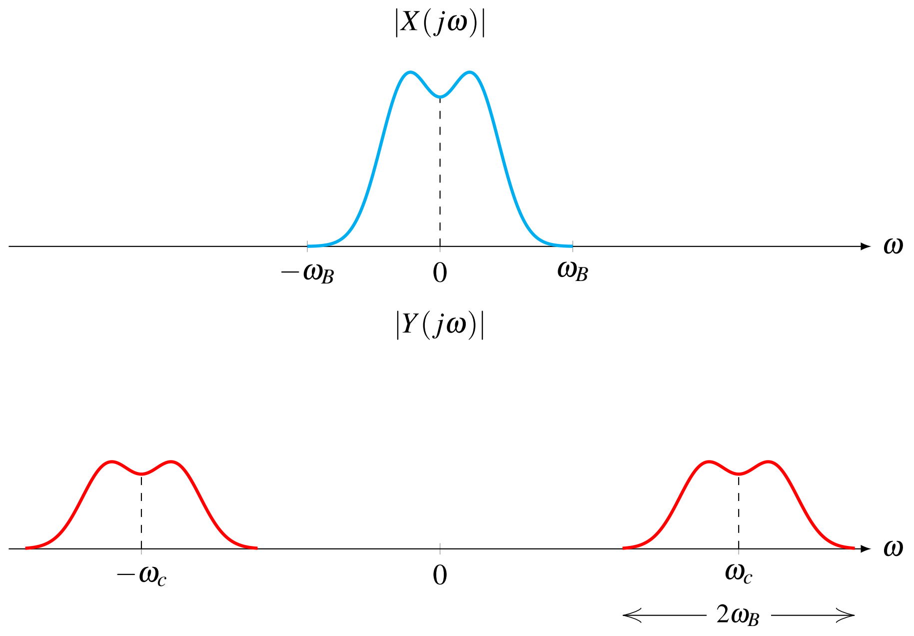
$$y(t) = Ax(t)\cos(\omega_c t) \tag{7.1}$$

, where $x(t)$ is the information signal and $\cos(\omega_c t)$ is the *carrier*. Observe that the "envelope" of $y(t)$ is $|x(t)|$ and *not* $x(t)$. So we cannot recover $x(t)$ from $y(t)$ using an "envelope detector."

7.2.1 Frequency Representation of Modulated Signals

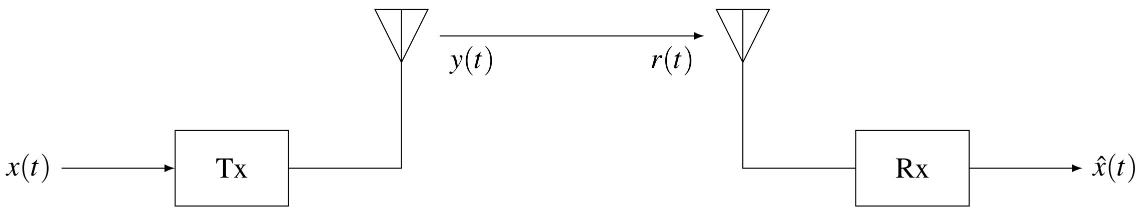
By the modulation property,

$$Y(j\omega) = \mathscr{F}\{y(t)\} = \frac{A}{2}\left[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))\right] \tag{7.2}$$



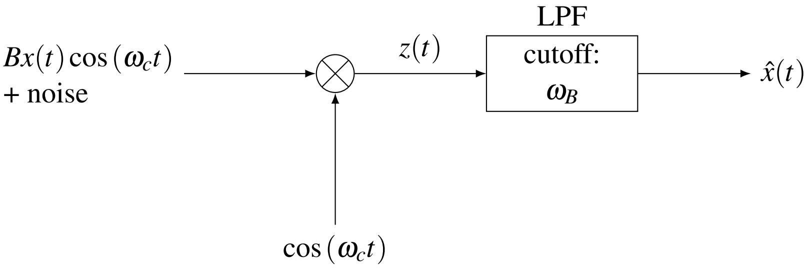
Note that the bandwidth of $y(t)$ is $2\omega_B$, i.e. we need twice as much bandwidth to transmit a SC AM signal compared to the original information signal $x(t)$.

7.3 Demodulation



$$\begin{aligned} y(t) &= Ax(t)\cos(\omega_c t) \\ r(t) &= \alpha y(t) + \text{ noise} \\ &= \underbrace{\alpha A}_{B} x(t)\cos(\omega_c t) + \text{ noise} \end{aligned}$$

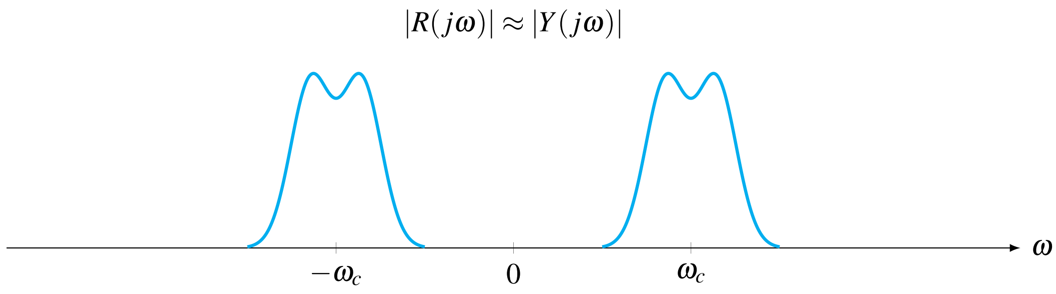
How can we recover $x(t)$ from $r(t)$?
Demodulation:

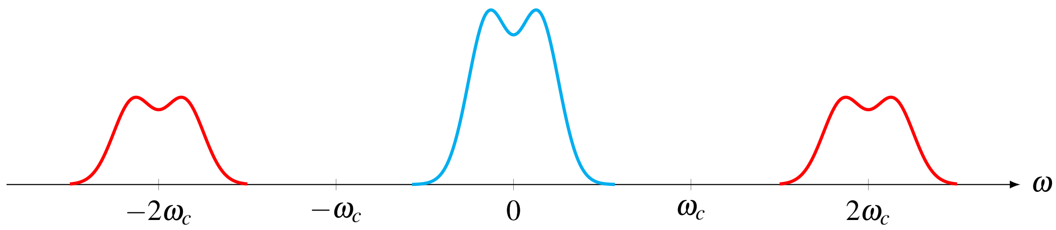


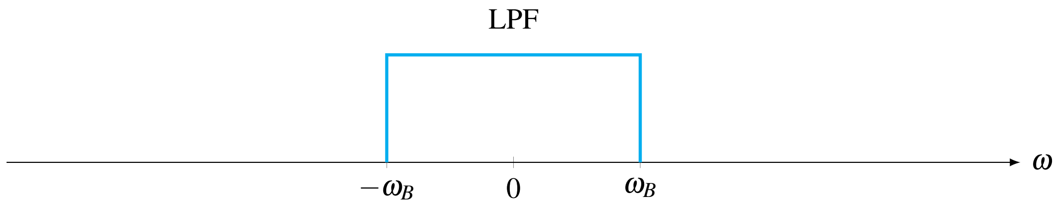
We'll see why this works both in the frequency domain AND in the time domain. For now we do not consider noise.

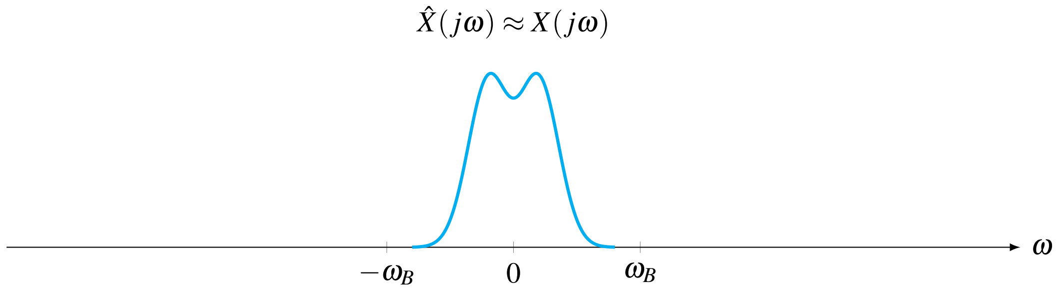
7.3.1 Frequency Domain Analysis

$$\begin{aligned} Z(j\omega) &= \mathscr{F}\{z(t)\} = \mathscr{F}\{r(t)\cos(\omega_c t)\} \\ &= \frac{1}{2}R(j(\omega-\omega_c)) + \frac{1}{2}R(j(\omega+\omega_c)) \end{aligned}$$


$|R(j\omega)| \approx |Y(j\omega)|$




LPF


$\hat{X}(j\omega) \approx X(j\omega)$

so the information signal is recovered!

7.3.2 Time Domain Analysis

$$\begin{aligned} z(t) = r(t)\cos(\omega_c t) &= Bx(t)\cancelto{\frac{1+\cos(2\omega_c t)}{2}}{\cos^2(\omega_c t)} \\ &= \underbrace{\frac{B}{2}x(t)}_{\substack{\text{centered}\\ \text{around 0}\\ \text{frequency}}} + \underbrace{\frac{B}{2}x(t)\cos(2\omega_c t)}_{\substack{\text{modulated with}\\ \text{a cos of } 2\omega_c\\ \text{so centered at}\\ \pm 2\omega_c}} \end{aligned}$$

and after low-pass filtering, only the 1st term will pass, so

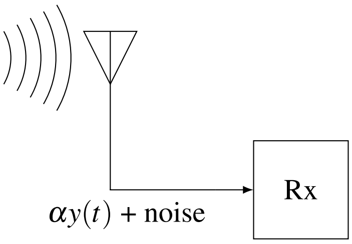
$$\hat{x}(t) = \frac{B}{2}x(t) \tag{7.3}$$

R Proportionality factors like $\frac{B}{2}$ are not that important in the analysis. Usually demodulation is followed by amplification (e.g. audio amplifier before driving the speakers).

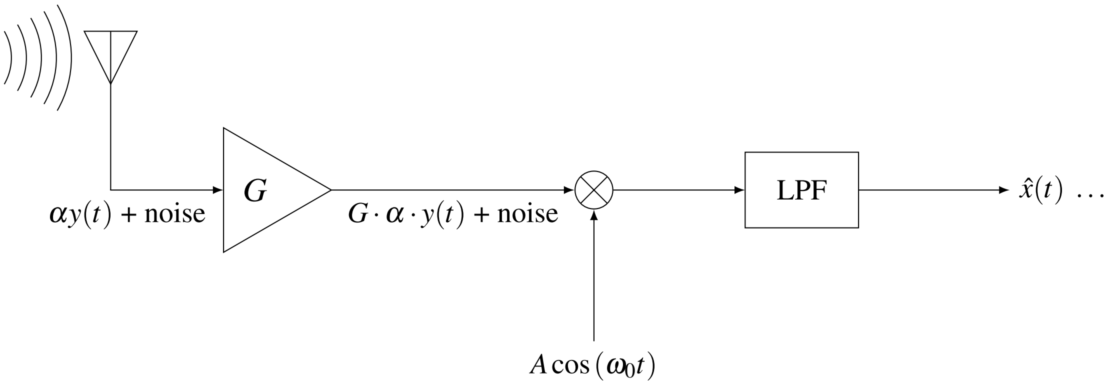
In reality,

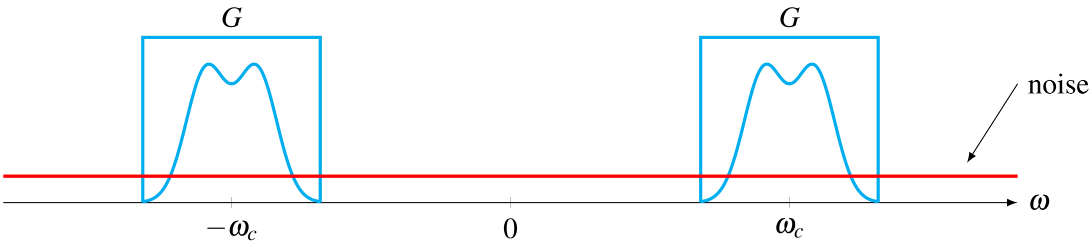
$$\hat{x}(t) = \frac{B}{2}x(t) + \text{ noise (the noise component through the LPF)}$$

R



- The signal picked up by the antenna is very low, usually on the order of $\sim \mu$Volts.
- So before any processing, we need to amplify it in order to bring it into the ∼Volts range.
- First stage of R_x is a high-gain amplifier, $G \approx 10^6$.





Really this is a high-gain bandpass filter.

There was a hidden assumption in the preceeding analysis: the carrier at the Tx $\cos(\omega_c t)$ and the cos at the Rx $\cos(\omega_c t)$ were ***perfectly synchronized***! What if there is a phase difference? Let's redo the analysis (time-domain).

$$
\begin{aligned}
y(t) &= Ax(t)\cos(\omega_c t)\\
\hat{x}(t) &= Bx(t)\cos(\omega_c t)\cos(\omega_c t+\phi)\\
&= Bx(t)\left[\frac{\cos\phi}{2}+\frac{\cos(2\omega_c t+\phi)}{2}\right]\\
&= \underbrace{\frac{B}{2}x(t)\cos\phi}_{\substack{\text{$\cos\phi$ is a}\\ \text{constant, so}\\ \text{this is}\\ \text{proportional}\\ \text{to $x(t)$}}} + \underbrace{\frac{B}{2}x(t)\cos(2\omega_c t+\phi)}_{\substack{\text{$x(t)$ modulated}\\ \text{with a $2\omega_c$}\\ \text{carrier so it is}\\ \text{centered at $2\omega_c$}}}
\end{aligned}
$$

So in reality, after low-pass filtering, we get:

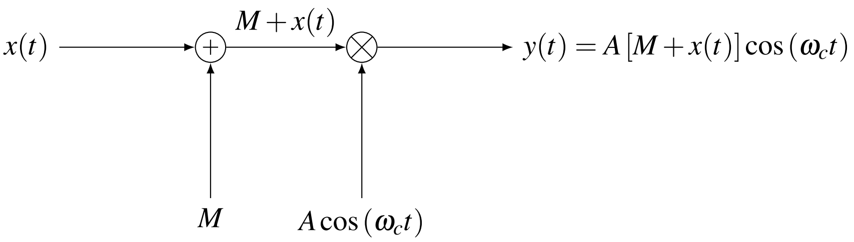
$$
\hat{x}(t) = \frac{B}{2}(\cos\phi)\cdot x(t) + \text{ noise (not affected by } \phi) \tag{7.4}
$$

- As ϕ varies slowly, when it gets close to $\phi \approx \frac{\pi}{2}$, then the signal part $\ll$ noise part.
- Any amount of amplification cannot solve the problem because it will amplify **BOTH** the signal and the noise.

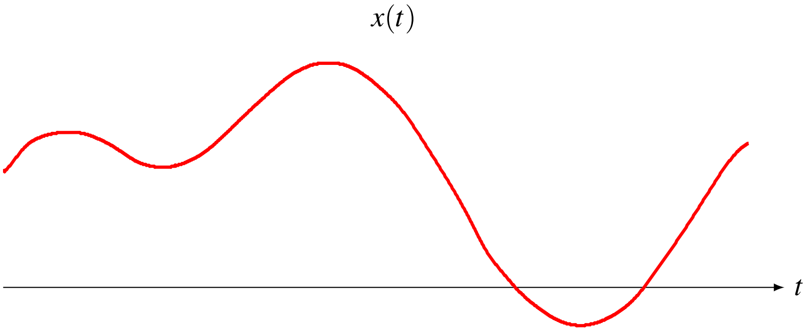
Two solutions:

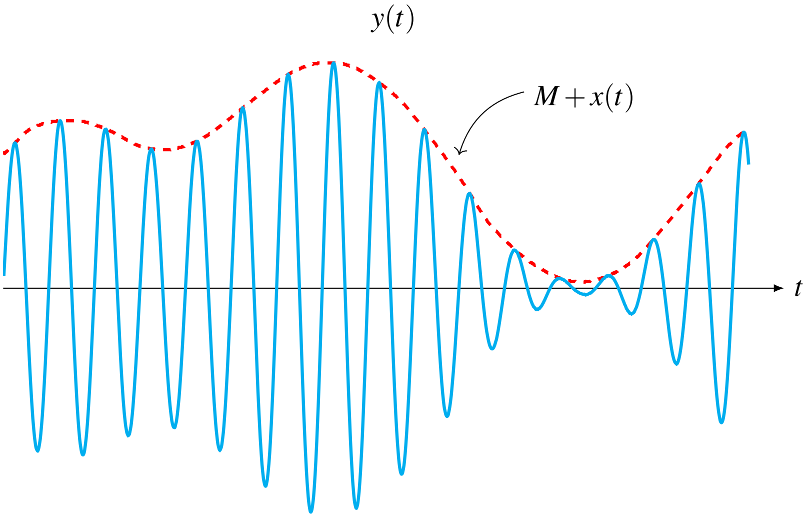
1. Devise a "synchronization" circuit that will "track" the incoming carrier phase and synchronize the local oscillator. This is called **"coherent" reception** (expensive, especially in the era of broadcast AM radio).
2. Change modulation scheme so that reception can be performed *non-coherently* $\Longrightarrow$ **Large-Carrier AM (LC-AM)**.

7.4 Large-Carrier AM (LC-AM)



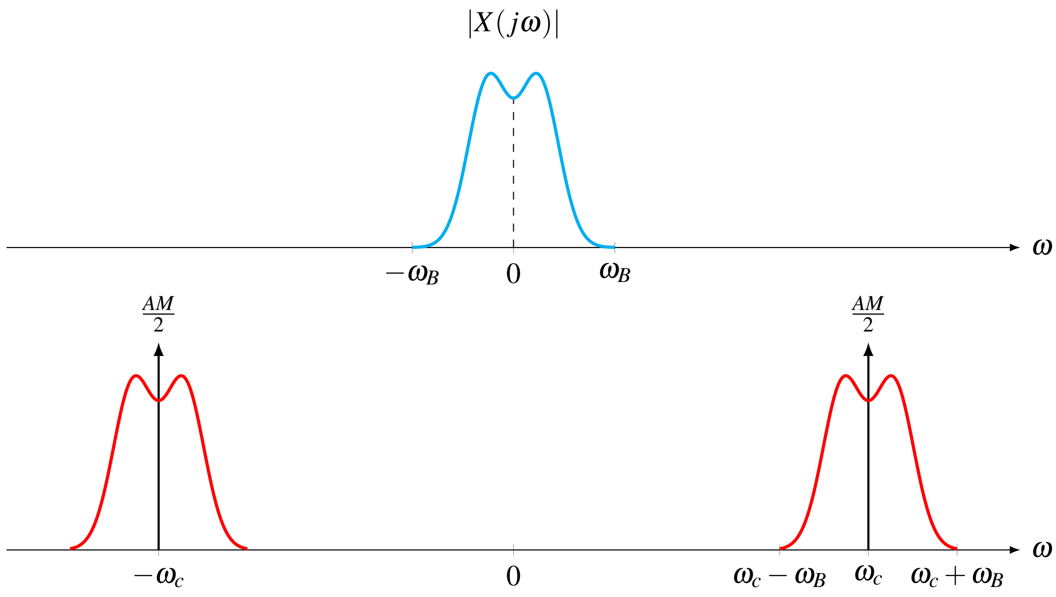
M is chosen so that $M \geq \max|x(t)|$ so $M+x(t) \geq 0$.





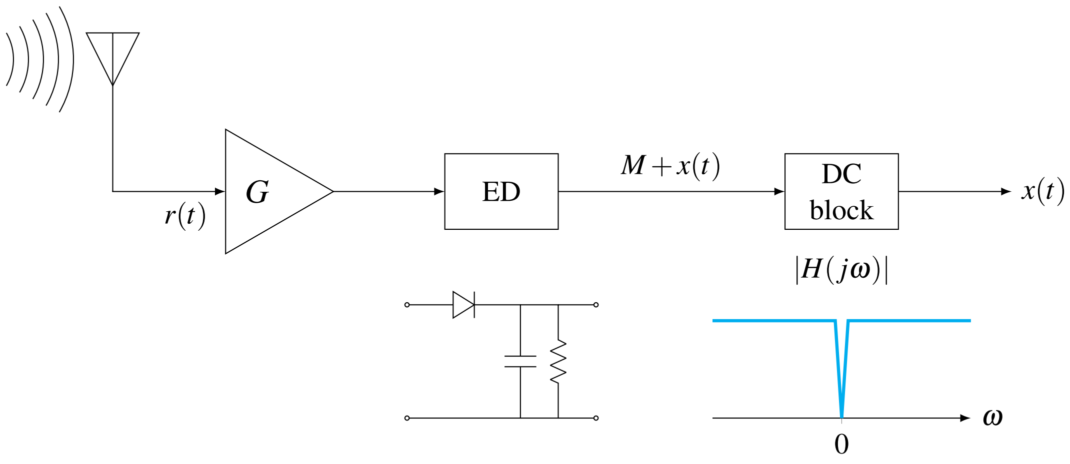
7.4.1 Frequency Representation

$$
\begin{array}{c}
y(t) = AM\cos\left(\omega_c t\right) + Ax(t)\cos\left(\omega_c t\right) \\
\qquad\qquad \downarrow \mathscr{F} \qquad\qquad\qquad\qquad \downarrow \mathscr{F} \\
\frac{AM}{2}\delta\left(\omega-\omega_c\right)+\frac{AM}{2}\delta\left(\omega+\omega_c\right)+\frac{A}{2}X\left(j\left(\omega-\omega_c\right)\right)+\frac{A}{2}X\left(j\left(\omega+\omega_c\right)\right)
\end{array}
$$



7.4.2 Demodulation

Since $M + x(t) > 0 \iff |M + x(t)| = M + x(t)$, the envelope of $y(t)$ is exactly $M + x(t)$. So reception can be performed using a single envelope detector:

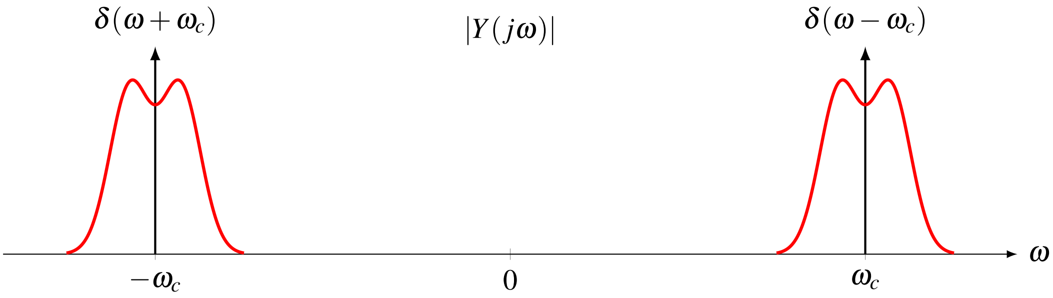


The DC block. cuts the DC component of the incoming signal. We assume that $x(t)$ does not have DC components (e.g. voice signal does not have audible components below some frequency).

Result: No need for expensive (coherent) receiver!

Drawback? Let's look at $y(t)$ again.

$$\begin{aligned} y(t) &= A\left[M + x(t)\right]\cos\left(\omega_c t\right) \\ &= AM\cos\left(\omega_c t\right) + Ax(t)\cos\left(\omega_c t\right) \\ &\qquad\qquad \downarrow \mathscr{F} \\ Y(j\omega) &= \frac{AM}{2}\left[\delta\left(\omega-\omega_c\right)+\delta\left(\omega+\omega_c\right)\right] + \frac{A}{2}\left[X\left(j\left(\omega-\omega_c\right)\right)+X\left(j\left(\omega+\omega_c\right)\right)\right] \end{aligned}$$



The part of $y(t)$ that equals $AM\cos\left(\omega_c t\right)$ does not carry any information. Yet we need power to transmit it:

$$P_c = \frac{(A\cdot M)^2}{2}$$

The information-carrying signal $Ax(t)\cos\left(\omega_c t\right)$ has power $P_I = \frac{A^2 P_x}{2}$ (why?). So the total power is:

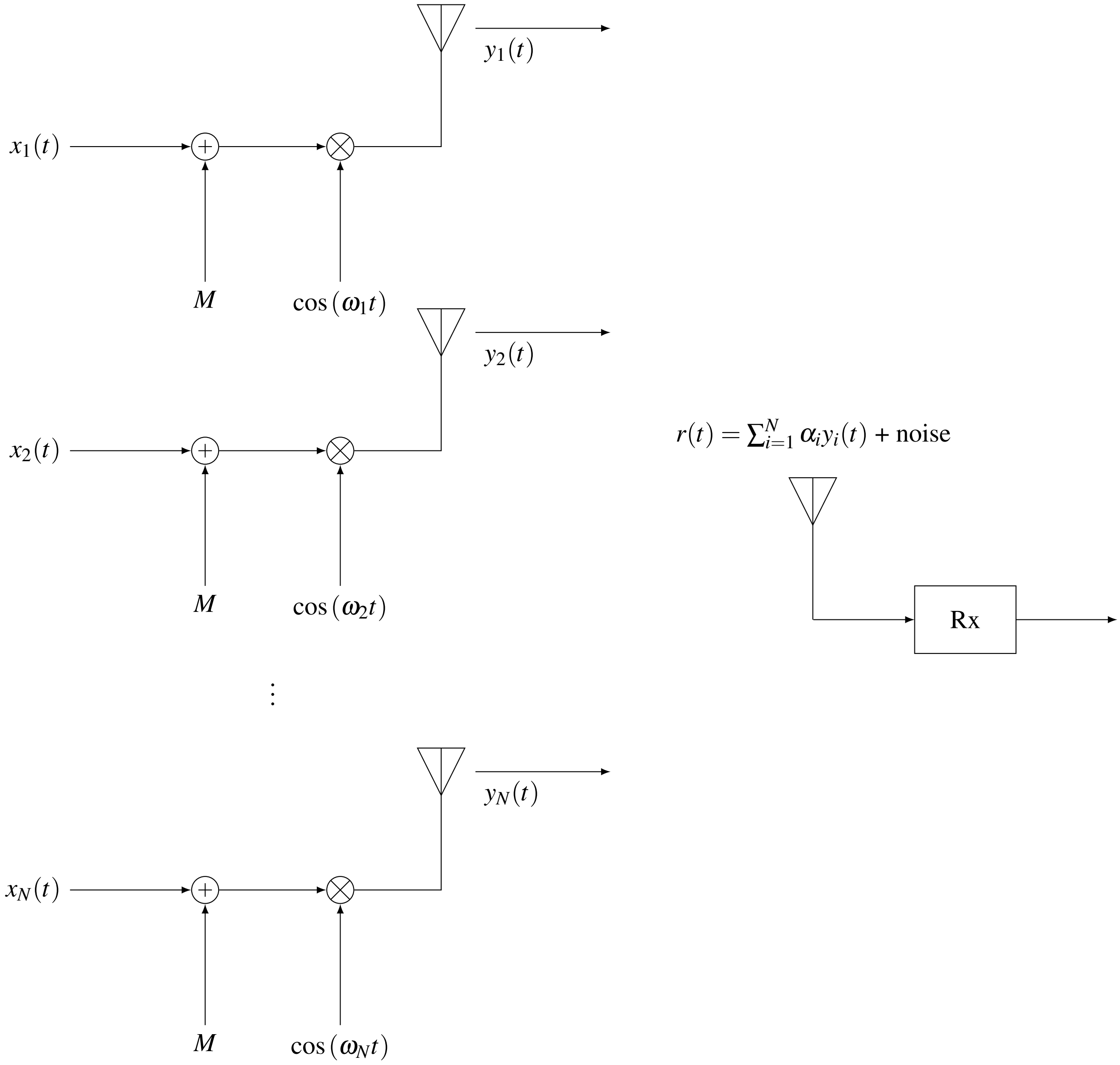
$$\begin{aligned} P_{tot} &= \frac{A^2}{2}\left(M^2 + P_x\right) \\ &= \frac{A^2}{2}P_x\left(1+\frac{M^2}{P_x}\right) \end{aligned}$$

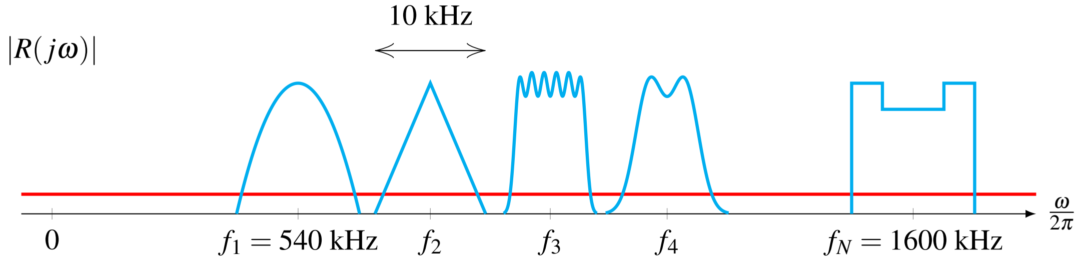
The $\left(1+\frac{M^2}{P_x}\right)$ is a factor of "wasted" power: it is only there to make $M + x(t) \geq 0$. So LC-AM is not as power-efficient as SC-AM (price we have to pay for simple demodulation).

7.5 Commercial AM Radio

107 stations, each station low-pass filters audio signals at 5 kHz and uses carrier frequency:

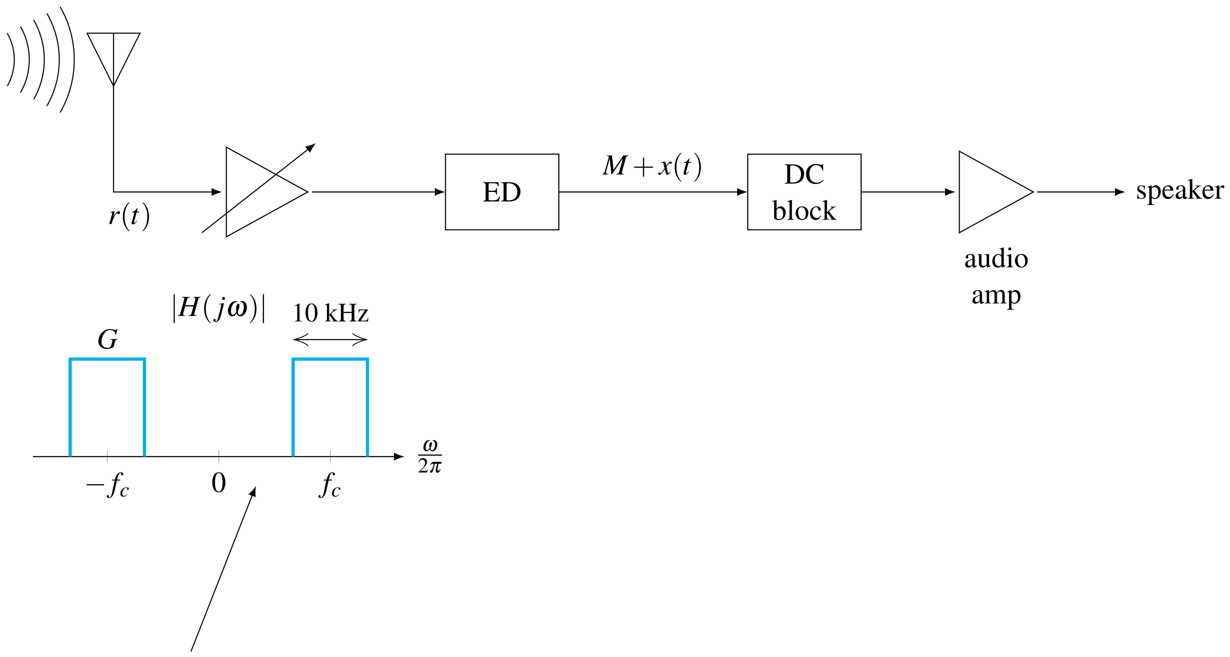
1	2	3	…	…	…	$N = 107$
540 kHz	550 kHz	560 kHz	…	…	…	1600 kHz





How does the Rx select the desired station?

- **Method 1:** Use a tunable filter/amplifier, followed by ED (envelope detector) and DC blocking element.



This needs to be a high-gain amplifier
μV $\to$ V, $G \approx 10^6$

f_c is tunable to any frequency (540 kHz, …, 1600 kHz).

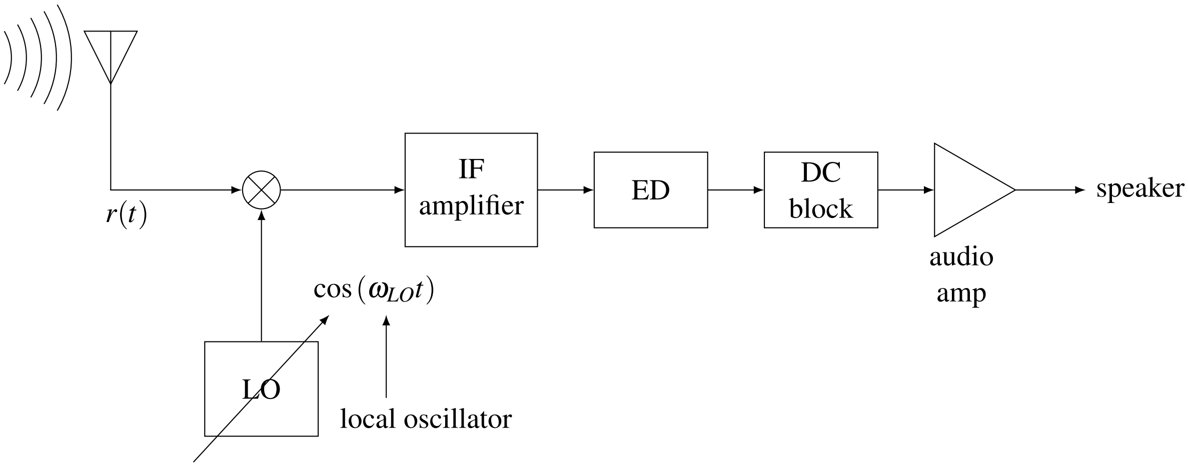
A measure of the amp cost is:

$$Q \approx \frac{f_c}{BW} \sim \frac{540\text{ kHz} \sim 1600\text{ kHz}}{10\text{ kHz}} \sim (54 \sim 160) \sim 100$$

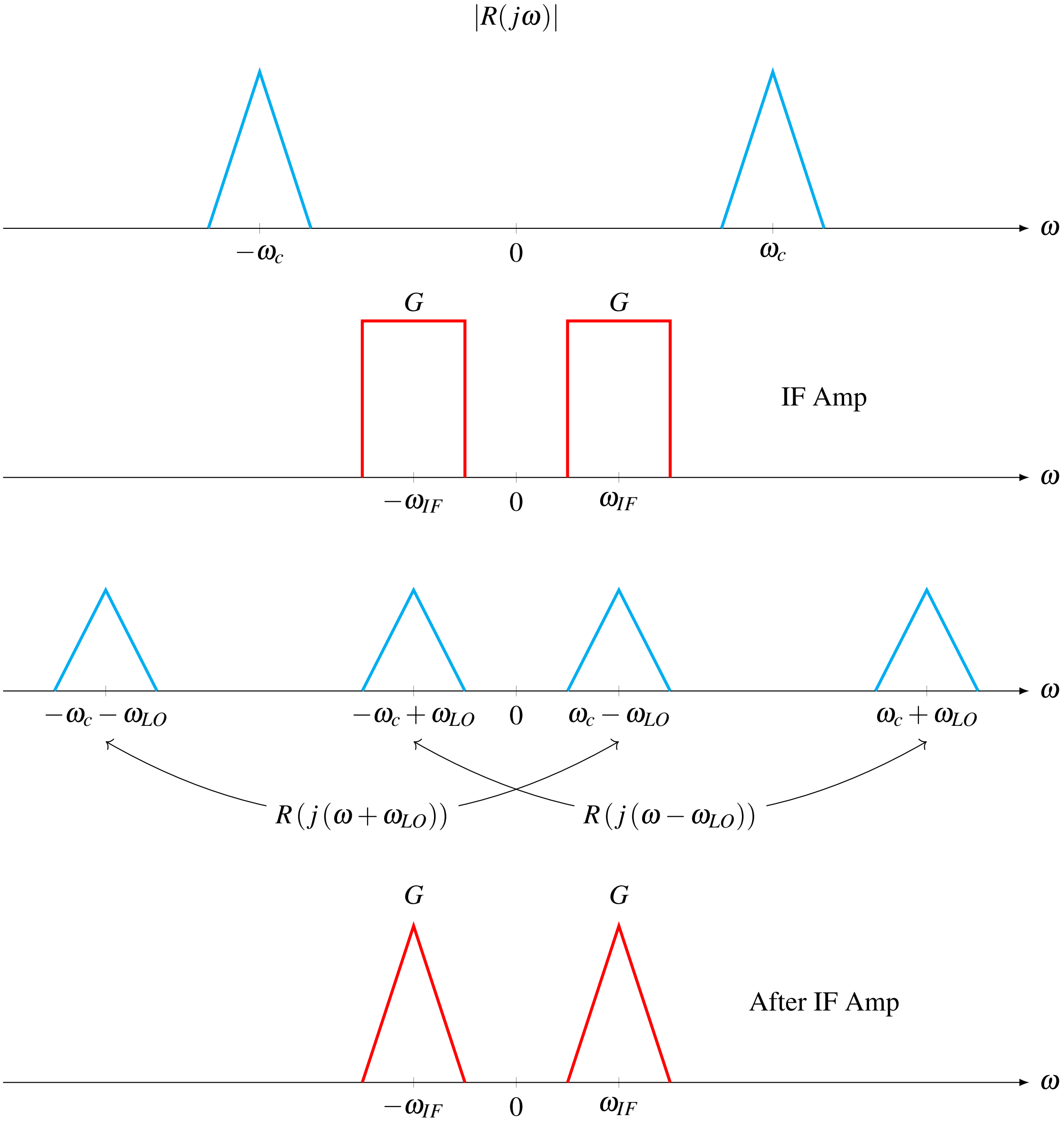
It is expensive to build a high-Q ($Q \approx 100$) *and* tunable amplifier.

7.5.1 Solution: Superheterodyne Receiver

Build a **SINGLE** high-Q amplifier at a constant frequency, and use modulation property of Fourier Transform to bring desired signal to that frequency. E.g., assume that we build an amplifier at $f_{IF} = 455$ kHz, where IF stands for Intermediate Frequency, and 455 kHz is the standard off-the-sell component. The superheterodyne receiver is:



How do we choose ω_{LO}? Recall the modulation property:



So we choose ω_{LO} such that:

$$\omega_c-\omega_{LO}=\omega_{IF} \iff \boxed{\omega_{LO}=\omega_c-\omega_{IF}} \tag{7.5}$$

$$\left(\begin{array}{l}\text{Alternatively, we can do:}\\ \omega_c-\omega_{LO}=-\omega_{IF} \iff \omega_{LO}=\omega_c+\omega_{IF}\end{array}\right)$$

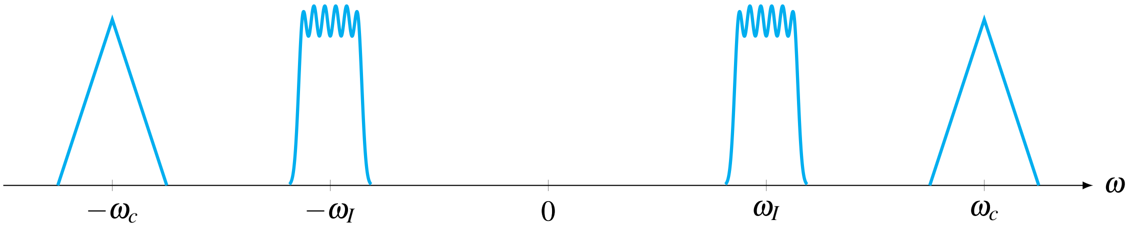
However, if $\omega_{IF}>\omega_c$, then

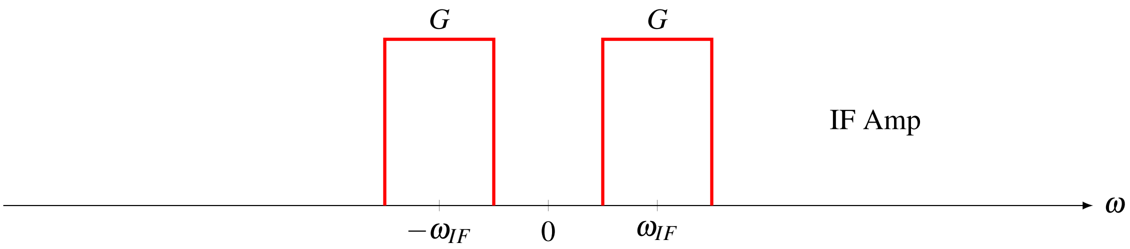
$$\omega_c+\omega_{LO}=\omega_{IF} \iff \omega_{LO}=\omega_{IF}-\omega_c \tag{7.6}$$

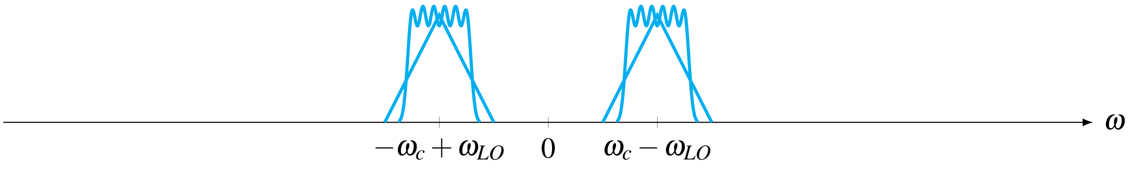
So in general, this says:

$$\omega_{LO}=|\omega_c-\omega_{IF}| \tag{7.7}$$

We solved one problem but we created another! There exists (possibly) another station that will interfere with our desired station.







Recall $\omega_c - \omega_{LO} = -\omega_{IF}$:

$$\omega_I - \omega_{LO} = -\omega_{IF} \iff \boxed{\omega_I = \omega_{LO} - \omega_{IF}} \tag{7.8}$$

$$\implies \omega_I = (\omega_c - \omega_{IF}) - \omega_{IF} \implies \boxed{\omega_I = \omega_c - 2\omega_{IF}} \tag{7.9}$$

If $\omega_{IF} > \omega_c \implies \omega_{LO} = \omega_{IF} - \omega_c$ and also

$$-\omega_I + \omega_{LO} = -\omega_{IF} \iff \boxed{\omega_I = \omega_{LO} + \omega_{IF} = 2\omega_{IF} - \omega_c} \tag{7.10}$$

so in general,

$$\omega_I = \underset{\substack{\uparrow \\ \text{due to symmetry}}}{|\omega_c - 2\omega_{IF}|} \tag{7.11}$$

■ **Example 7.1** With $\omega_{IF} = 455$ kHz, $2\omega_{IF} = 910$ kHz, and

$$\begin{array}{rcccccccc} \omega_c = & 540, & 550, \ldots, & 1440, & 1450, & 1460, & \ldots, & 1600 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ \omega_I = & +370, & +360, \ldots, & 530, & 540, & 550, & \ldots, & 690 \end{array}$$

There are no stations at +370, +360, …, and 530, 540, 550, …, 690 are the interfering stations. Thus for desired stations in the upper part of the band, we have this interference problem (interfering station is called "image" station).

Alternatively, if we had chosen ω_{LO} such that $\omega_c - \omega_{LO} = -\omega_{IF} \iff \omega_{LO} = \omega_c + \omega_{IF}$, the image is at $\omega_I - \omega_{LO} = \omega_{IF} \iff \omega_I = \omega_c + 2\omega_{IF}$. In this case the lower stations would have an image:

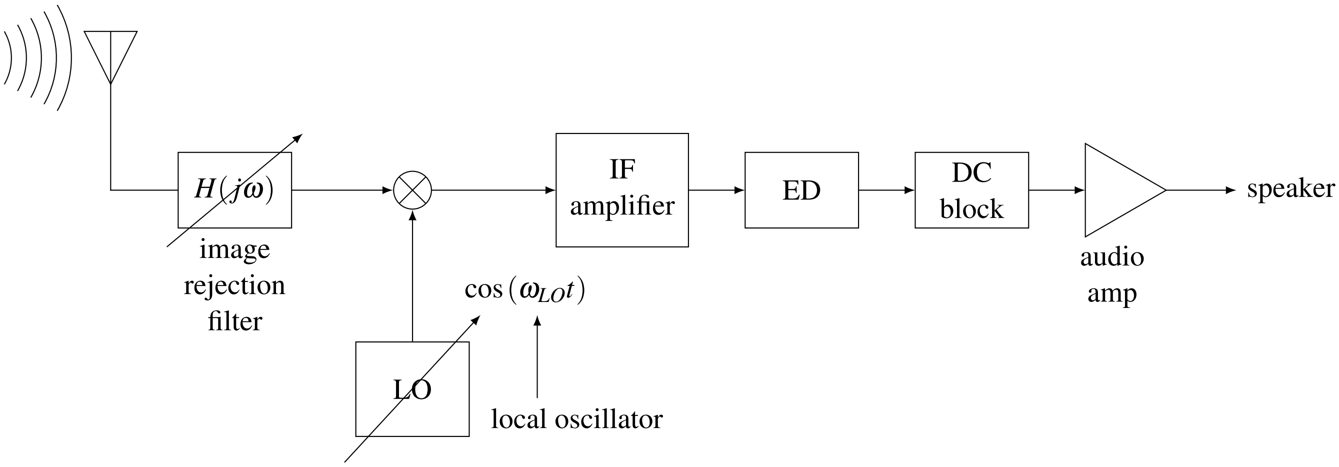
$$\begin{array}{cc}
\omega_c = 540 & \omega_I = 1450 \\
\vdots & \vdots \\
690 & 1600 \\
\hline
700 & \cancel{1610} \\
\vdots & \vdots \\
1600 & \cancel{2510}
\end{array}$$

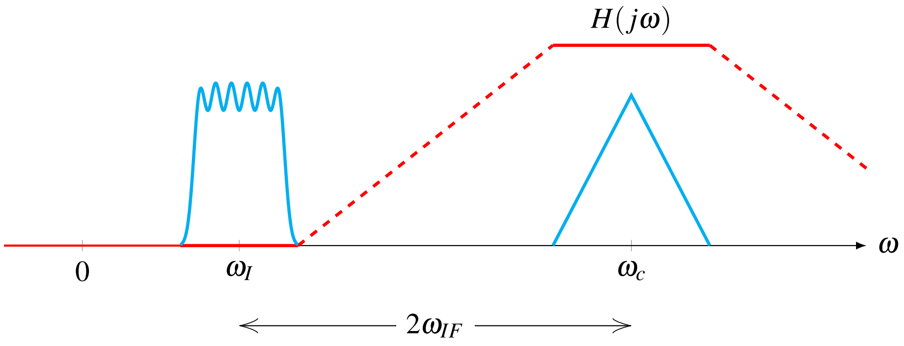
Overall summary: (given ω_c, ω_{IF})

$$\begin{array}{c|c}
\omega_{LO} & \omega_I \\
\hline
\begin{cases} \omega_c - \omega_{IF} & (\omega_c > \omega_{IF}) \\ \omega_{IF} - \omega_c & (\omega_{IF} > \omega_c) \end{cases} & \begin{array}{c} \omega_c - 2\omega_{IF} \\ 2\omega_{IF} - \omega_c \end{array} \\
|\omega_c - \omega_{IF}| & |\omega_c - 2\omega_{IF}| \\
\hline
\omega_c + \omega_{IF} & \omega_c + 2\omega_{IF}
\end{array}$$

■

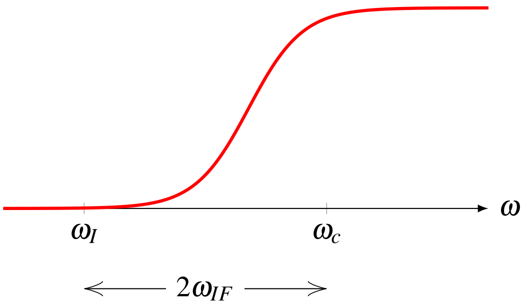
Solution to the "image" problem: Filter out the image station ***before*** mixing:





$$|H(j\omega)| = \begin{cases} 1 & |\omega - \omega_c| < 2\pi \cdot 5\text{ kHz} \\ 0 & |\omega - \omega_I| < 2\pi \cdot 5\text{ kHz} \\ \text{arbitrary} & \text{else} \end{cases} \tag{7.12}$$

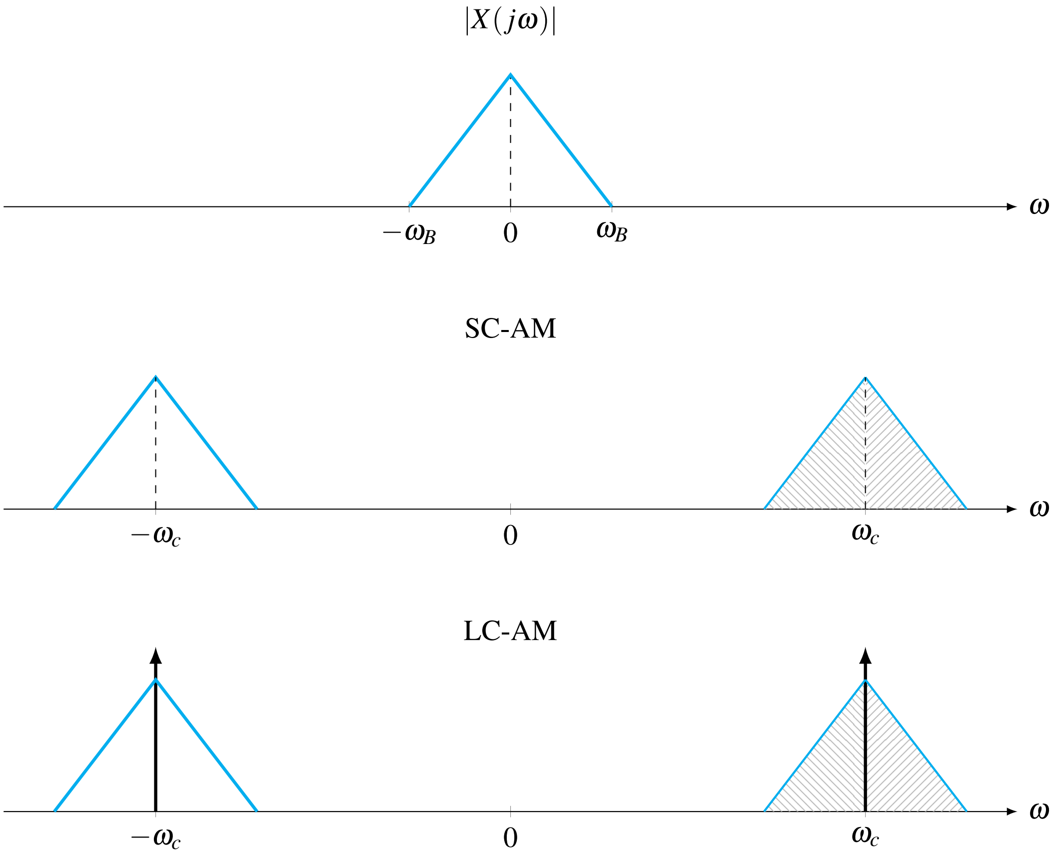
This is a tunable but inexpensive filter.
Tunable: $f_c = 1450, \ldots, 1600$ kHz



$$Q \approx \frac{1450 \sim 1600}{910} \approx 1.5$$

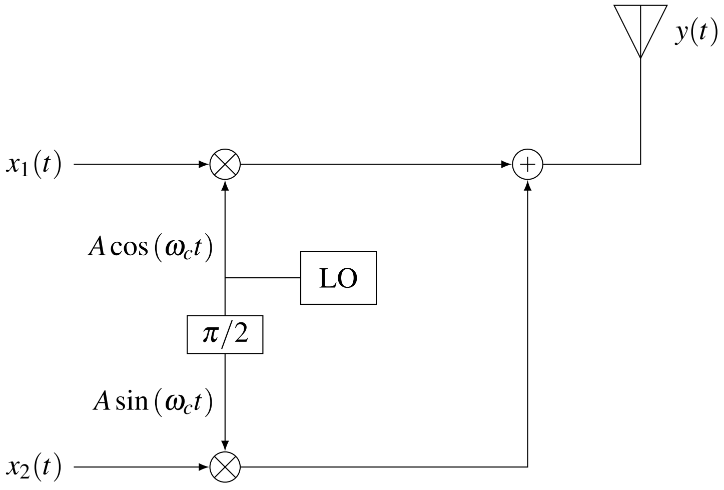
7.6 Quadrature and Single Sideband Amplitude Modulation

We have discussed SC-AM and LC-AM. They both require $BW = 2\omega_B$ for transmission of a signal with $BW = \omega_B$.

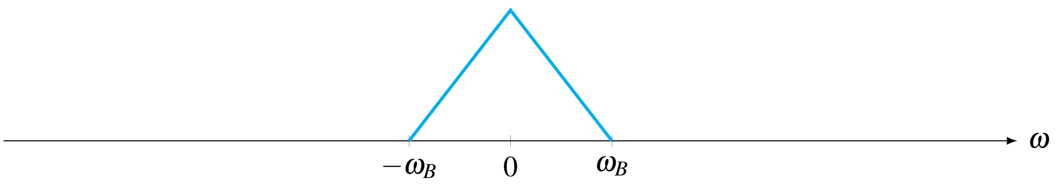


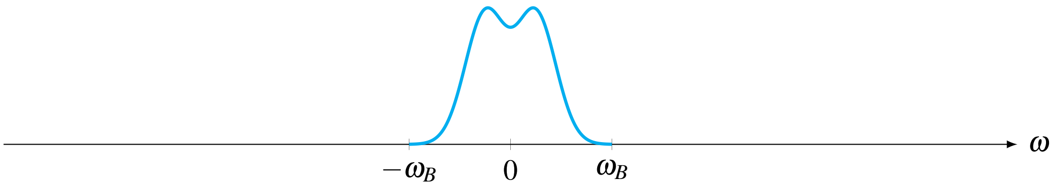
However the lower and upper sidebands of the modulated signal contain the same information (for the case of real $x(t)$). So we are "wasting" half of the bandwidth. How can we economize?

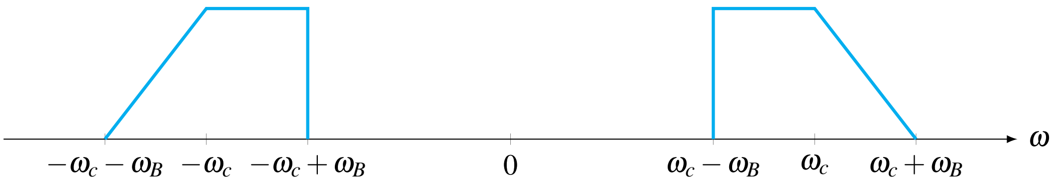
- ***Method 1:*** Quadrature-AM
  Idea: Transmit two information signals $x_1(t)$ and $x_2(t)$ on the same band!



$$
\begin{aligned}
y(t) =& Ax_1(t)\cos(\omega_c t) + Ax_2(t)\sin(\omega_c t)\\
Y(j\omega) =& AX_1(j\omega) * \left[\frac{1}{2}\delta(\omega-\omega_c) + \frac{1}{2}\delta(\omega+\omega_c)\right] + AX_2(j\omega) * \left[\frac{1}{2j}\delta(\omega-\omega_c) - \frac{1}{2j}\delta(\omega+\omega_c)\right]\\
=& \frac{A}{2}X_1(j(\omega-\omega_c)) + \frac{A}{2}X_1(j(\omega+\omega_c))\\
&+ \underbrace{\frac{A}{2j}X_2(j(\omega-\omega_c))}_{\text{both at } \omega_c} - \underbrace{\frac{A}{2j}X_2(j(\omega+\omega_c))}_{\text{both at } -\omega_c}
\end{aligned}
$$

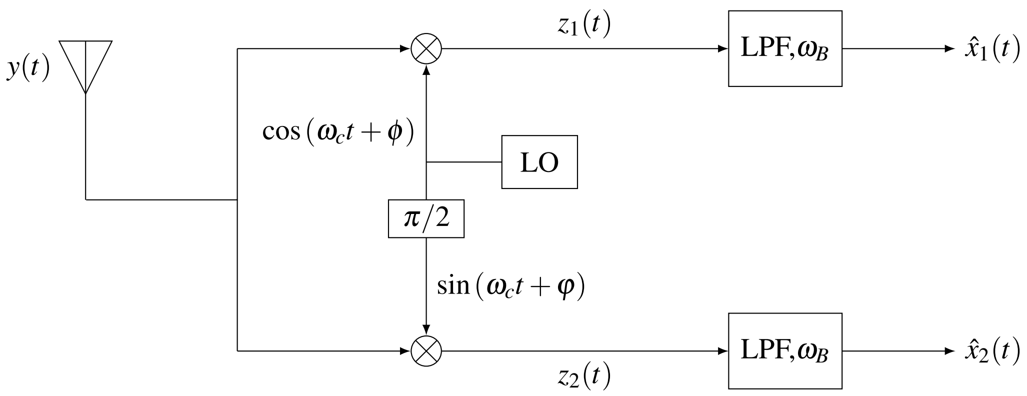

$X_1(j\omega)$


$X_2(j\omega)$


$Y(j\omega)$

So now we need $2\omega_B$ bandwidth for two stations so ω_B per station! But can we recover them at the Rx?

7.6.1 Coherent Demodulation



$$\begin{aligned}
z_1(t) &= y(t)\cos(\omega_c t+\phi)\\
&= [Ax_1(t)\cos(\omega_c t)+Ax_2(t)\sin(\omega_c t)]\cos(\omega_c t+\phi)\\
&= Ax_1(t)\cos(\omega_c t)\cos(\omega_c t+\phi)+Ax_2(t)\sin(\omega_c t)\cos(\omega_c t+\phi)\\
&= Ax_1(t)\frac{\cos\phi+\cos(2\omega_c t+\phi)}{2}+Ax_2(t)\frac{-\sin\phi+\sin(2\omega_c t+\phi)}{2}\\
&= \frac{A}{2}[x_1(t)\cos\phi-x_2(t)\sin\phi]\\
&\quad \left.\begin{array}{l}\frac{A}{2}x_1(t)\cos(2\omega_c t+\phi)\\ \frac{A}{2}x_2(t)\sin(2\omega_c t+\phi)\end{array}\right\}\ \text{modulated at } 2\omega_c \text{ so are out by LPF}
\end{aligned}$$

so

$$\hat{x}_1(t)=\frac{A}{2}x_1(t)\cos\phi-\frac{A}{2}x_2(t)\sin\phi \tag{7.13}$$

Similarly you can find that

$$\begin{aligned}
z_2(t) &= y(t)\sin(\omega_c t+\varphi)\\
&= [Ax_1(t)\cos(\omega_c t)+Ax_2(t)\sin(\omega_c t)]\sin(\omega_c t+\varphi)\\
&= Ax_1(t)\frac{\sin\phi+\sin(2\omega_c t+\phi)}{2}+Ax_2(t)\frac{\cos\phi-\cos(2\omega_c t+\phi)}{2}
\end{aligned}$$

and after low-pass filtering,

$$\hat{x}_2(t)=\frac{A}{2}x_1(t)\sin\varphi+\frac{A}{2}x_2(t)\cos\varphi \tag{7.14}$$

So if we have **PERFECT** synchronization (i.e. $\varphi=0$), then:

$$\hat{x}_1(t)=\frac{A}{2}x_1(t)\sim x_1(t) \tag{7.15}$$

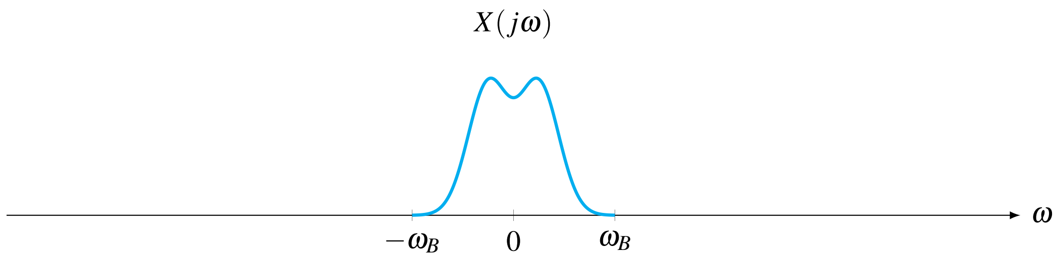
$$\hat{x}_2(t)=\frac{A}{2}x_2(t)\sim x_2(t) \tag{7.16}$$

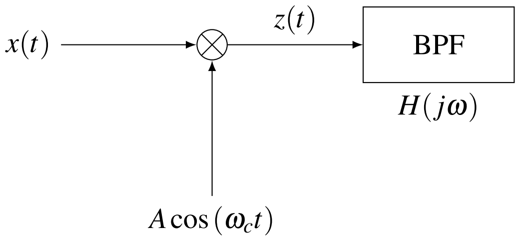
Otherwise we have cross-talk!

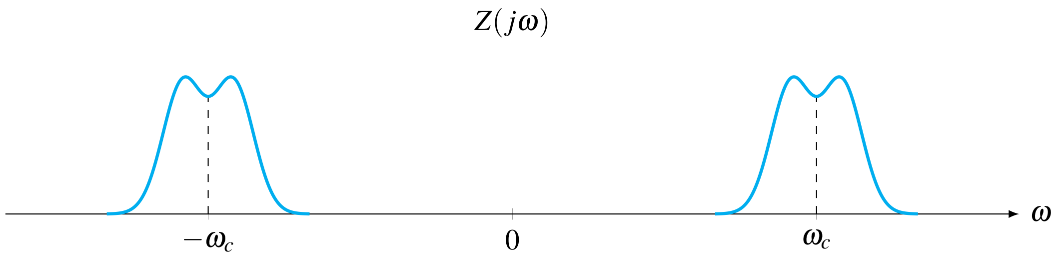
$$SINR_1 = \text{signal-to-interference \& noise-ratio} = \frac{P_1 \cos^2 \varphi}{P_2 \sin^2 \varphi + P_N}$$

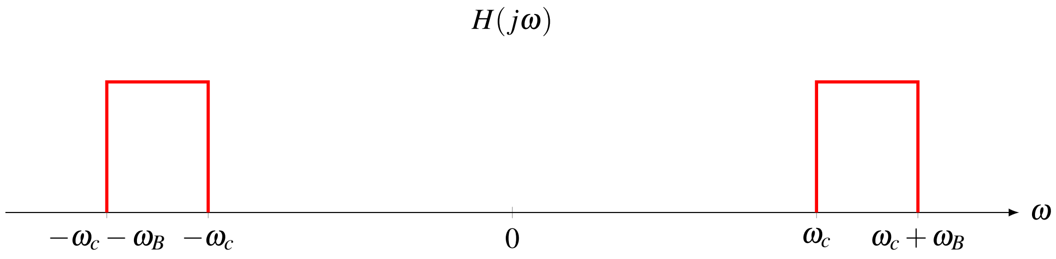
DEMO: GNURADIO QAM-MOD

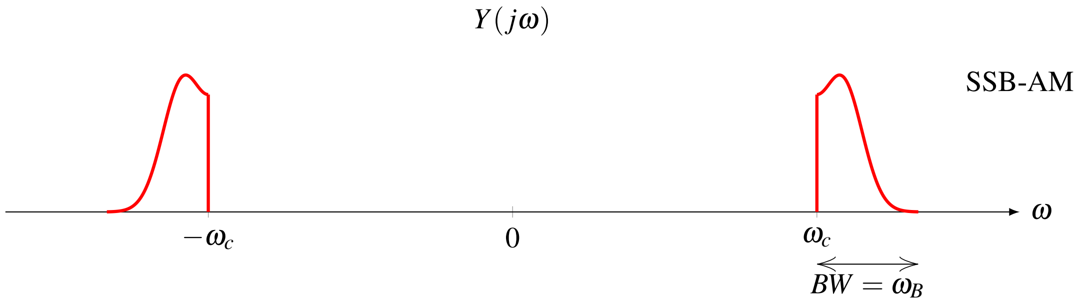
- ***Method 2***: Single Sideband AM (SSB-AM)
  Idea: Filter-out one of the sidebands before transmission



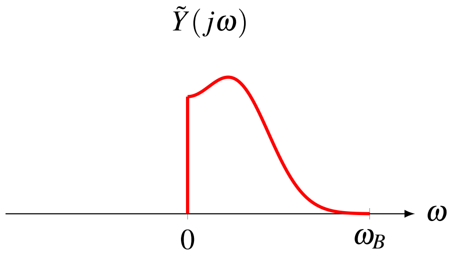






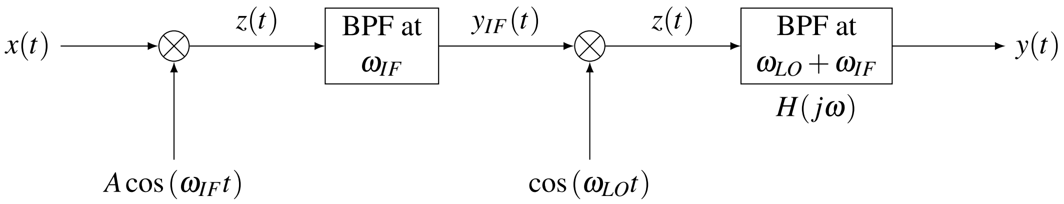


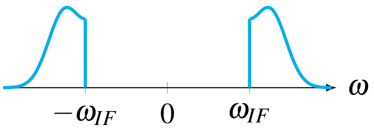
- It is difficult to give a time-domain expression for $y(t)$, because we do not know how to express the **complex** signal $\tilde{y}(t)$ with $\tilde{Y}(j\omega)$.

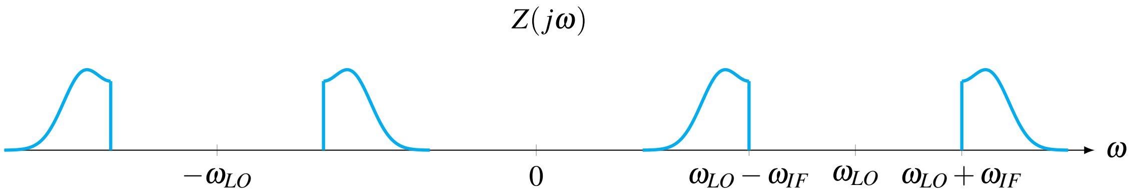


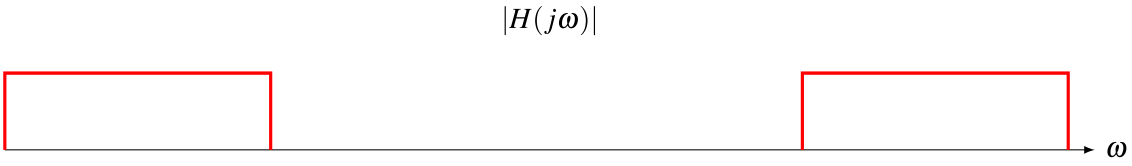
(we need the "Hilbert Transform" for that).

- Usually, this filtering is done in multiple IF stages, e.g.

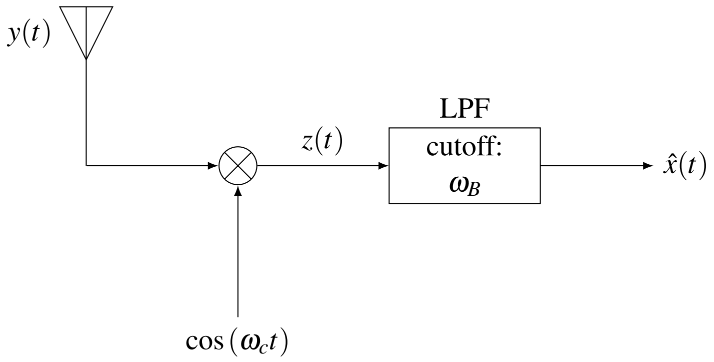


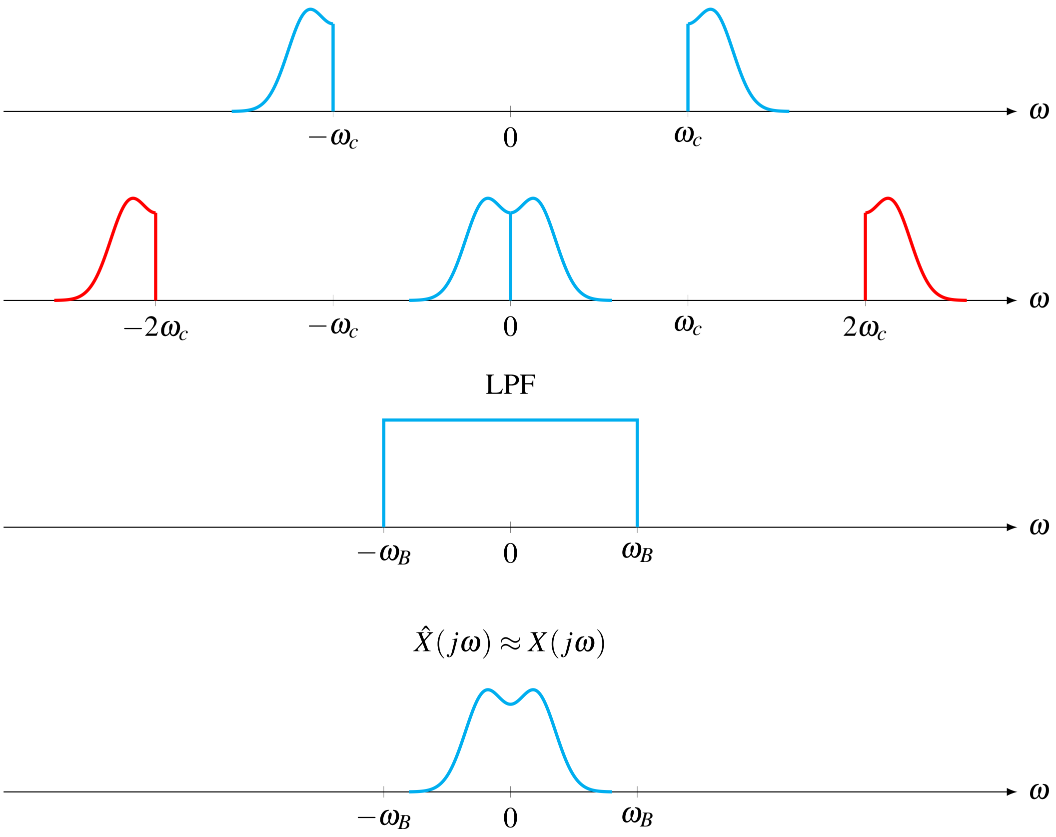






At the receiver demodulation is done exactly as in SC-AM:



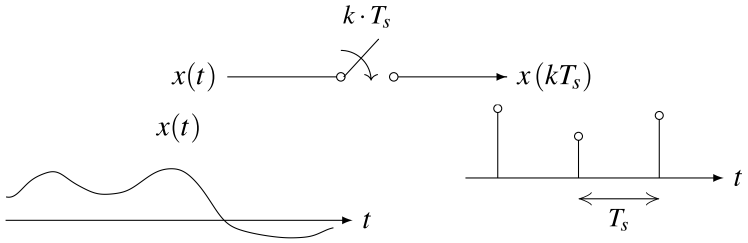


- What happens if we have a phase error?
- Recall: in SC-AM the effect was that signal power way reduced the SNR (signal-to-noise ratio) deteriorated.
- In SSB-AM there is an additional effect: signal is distorted.
  DEMO: GNURADIO

8. Sampling

Motivation: Every digital communication/storage system communicates/stores bits of information. This means that the original continuous-time information signal $x(t)$ (e.g. audio) has to be translated to bits. The first task in this process is "sampling."

8.1 Sampling

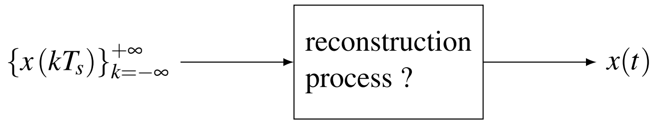


Definition 8.1 — Sampling. ***Sampling:*** retain only the "samples":

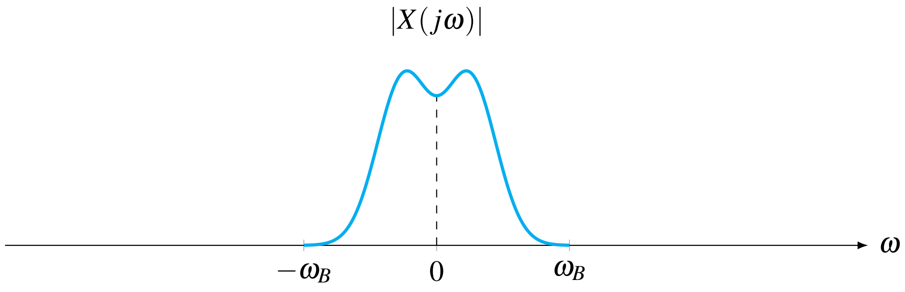
$$\begin{aligned} &\ldots x(-2T_s),\ x(-T_s),\ x(0),\ x(T_s),\ x(2T_s),\ \ldots \\ &= \{x(kT_s)\}_{k=-\infty}^{+\infty}\ (k \text{ integer}) \end{aligned} \tag{8.1}$$

from the continuous-time signal $x(t)$. T_s is the sampling period.

We will show that under certain conditions on $x(t)$ and T_s we can perfectly recover $x(t)$ from its infinitely-many samples $\{x(kT_s)\}_{k=-\infty}^{+\infty}$.

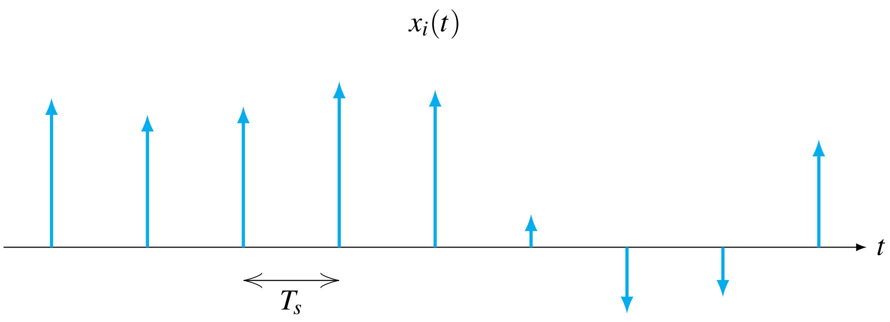


Let's assume that $x(t)$ is strictly bandlimited, i.e. it has absolute $BW = \omega_B$ and $X(j\omega)$ is:



Let's construct the following signal:

$$x_i(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t - kT_s) \tag{8.2}$$



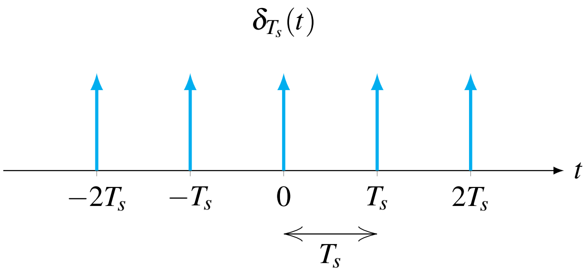
$$(8.2) \implies x_i(t) = \sum_{k=-\infty}^{+\infty} x(t)\delta(t-kT_s) \quad \text{(sampling property of } \delta\text{)}$$
$$= x(t)\cdot\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)$$
$$= x(t)\underbrace{\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)}_{\delta_{T_s}(t)}$$

We call $\delta_{T_s}(t)$ the "**delta train**" or "**delta comb**."

$$\begin{aligned} X_i(j\omega) &= \mathscr{F}\{x_i(t)\} = \mathscr{F}\{x(t)\delta_{T_s}(t)\} \\ &= \frac{1}{2\pi}X(j\omega) * \mathscr{F}\{\delta_{T_s}(t)\} \end{aligned} \tag{8.3}$$

8.1.1 *Paranthesis:* Let's evaluate $\mathscr{F}\{\delta_{T_s}(t)\}$

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{+\infty}\delta(t-kT_s)$$



$\delta_{T_s}(t)$ is periodic with period T_s, so we can express it as a Fourier Series:

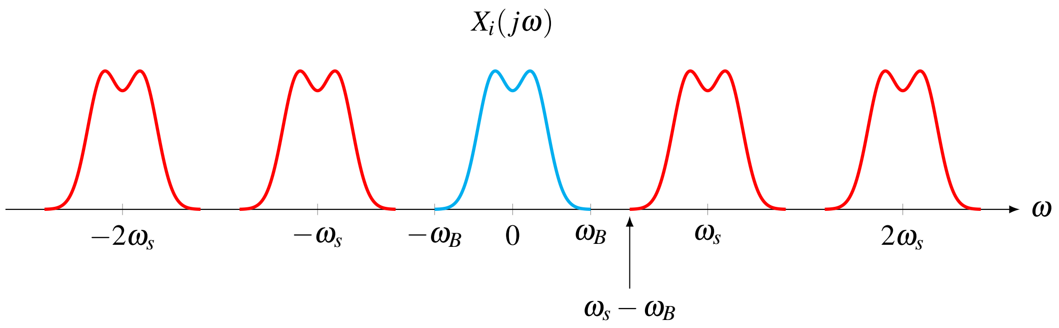
$$\delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_s t}, \quad \omega_s = \frac{2\pi}{T_s} \text{ (sampling frequency)}$$
$$c_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t)e^{-jn\omega_s t}dt = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)e^{-jn\omega_s t}dt = \frac{1}{T_s}$$

so

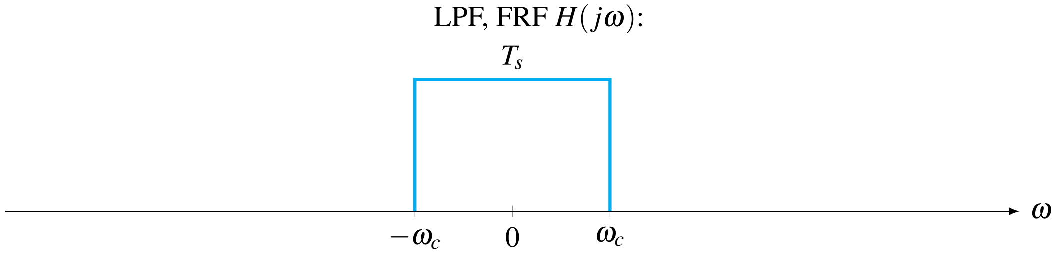
$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} e^{jn\omega_s t} \tag{8.4}$$

and

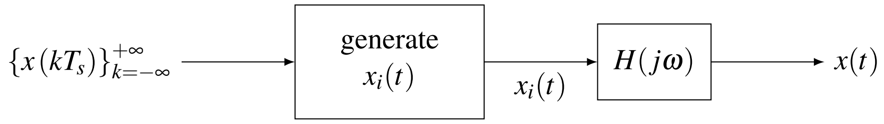
$$\begin{aligned}
\mathscr{F}\left\{\delta_{T_s}(t)\right\} &= \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} \mathscr{F}\left\{e^{jn\omega_s t}\right\} \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} 2\pi\delta\left(\omega - n\omega_s\right) && (8.5) \\
(8.3),(8.5) \implies X_i(j\omega) &= \frac{1}{2\pi}X(j\omega) * \left[\frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}\delta\left(\omega - n\omega_s\right)\right] \\
&\underset{\text{of conv.}}{\overset{\text{linearity}}{=}} \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} X(j\omega) * \delta\left(\omega - n\omega_s\right) \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} X\left(j\left(\omega - n\omega_s\right)\right) && (8.6)
\end{aligned}$$



It should now be clear how to recover $x(t)$ from $x_i(t)$: Lowpass filter $x_i(t)$ with an ideal filter with cutoff frequency $\omega_c \in (\omega_B, \omega_s - \omega_B)$.



Overall reconstruction device:



For the above operation to work we need:

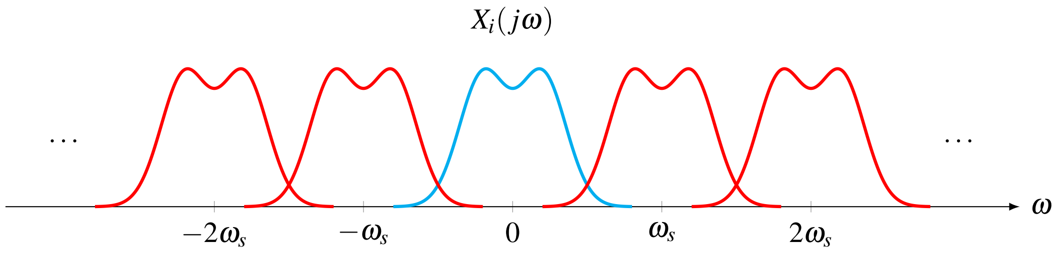
$$\omega_B < \omega_s - \omega_B \iff \boxed{\omega_s > 2\omega_B} \tag{8.7}$$

We define the ***Nyquist*** rate to be twice the signal bandwidth (i.e. $2\omega_B$).

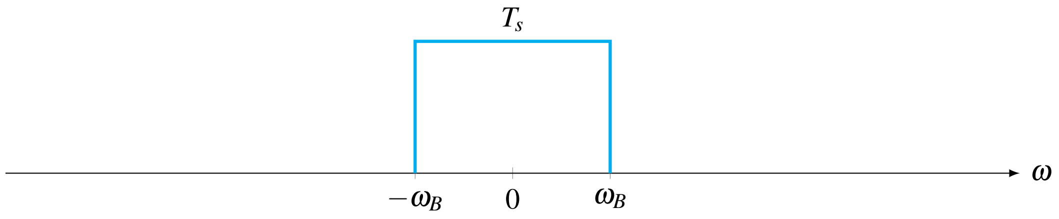
To summarize:

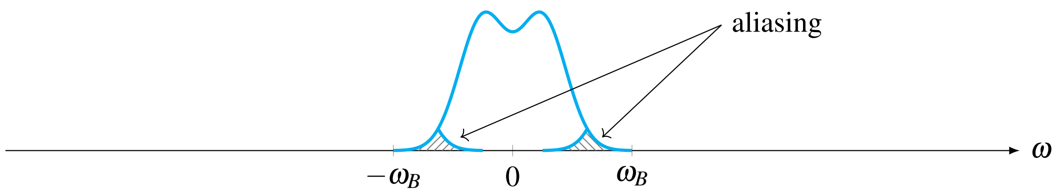
> For a signal $x(t)$ with absolute $BW = \omega_B$, if we sample with frequency $\omega_s = \frac{2\pi}{T_S} > 2\omega_B$, we can perfectly reconstruct it from its samples.

If $\omega_s < 2\omega_B$ the picture of $X_i(j\omega)$ is:

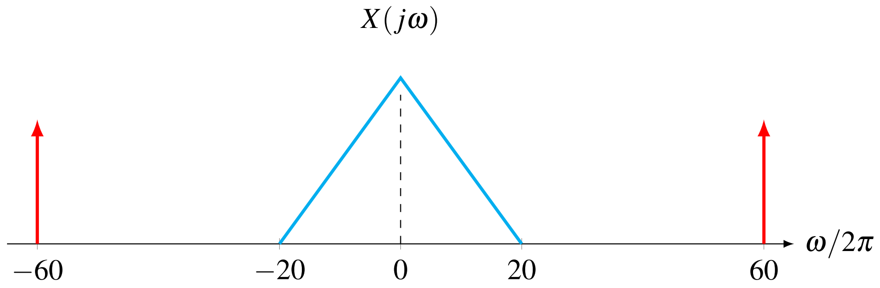


and so we cannot reconstruct $x(t)$ perfectly. Indeed if we use a lowpass filter with $\omega_c = \omega_B$

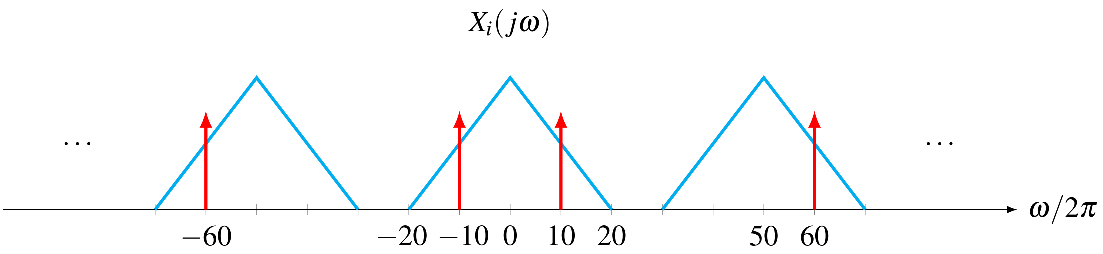




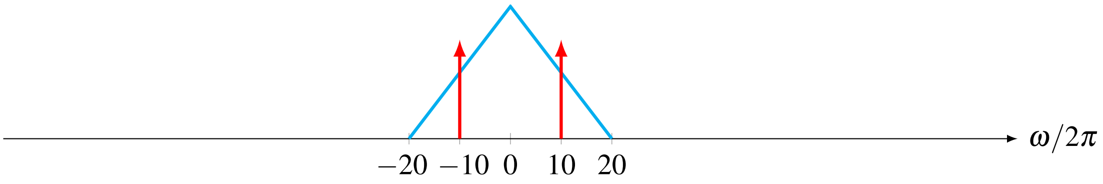
■ **Example 8.1** **Aliasing out-of-band interference:** Suppose a useful signal $x(t)$ has bandwidth 20 Hz and a sinusoidal interference is present at 60 Hz:



We sample this signal at 50 Hz ($> 2 \cdot 20 = 40$). The ideally sampled signal $x_i(t)$ looks like:



and after low-pass-filtering at 20 Hz we get:



So out-of-band interference (at 60 Hz) appears inside the useful band now (at 10 Hz). The reason for this is that the overall signal (desired signal + interference) has $BW = 60$ Hz and so it was undersampled (Nyquist freq. $= 120$ Hz) so there is aliasing.

Solutions:

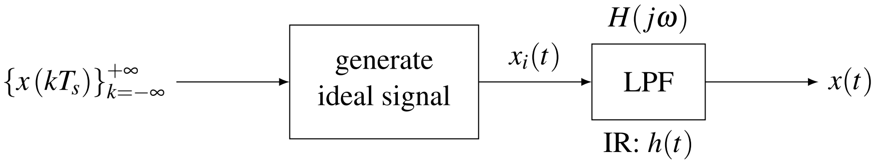
(a) Either filter out interference **before** sampling, or

(b) Sample with at least 120 samples/sec so no aliasing occurs.

■

8.2 Whitaker-Shannon Sampling Formula

Let us develop an explicit expression for $x(t)$ based on $\{x(kT_s)\}_{k=-\infty}^{+\infty}$ (interpolation formula). We know that under the assumption $\omega_s > 2\omega_B$ (and using an ideal low-pass filter with cutoff $\omega_c \in (\omega_B, \omega_s - \omega_B)$), we can perfectly reconstruct $x(t)$, i.e.



$$x(t) = x_i(t) * h(t) = \left[\sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s)\right] * h(t)$$

$$\implies \boxed{x(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,h(t-kT_s)} \quad \text{(this is an "interpolation" formula)} \tag{8.8}$$

Specifically for

$$H(j\omega) = \begin{cases} T_s & |\omega| < \omega_c \\ 0 & \text{else} \end{cases}$$

we have

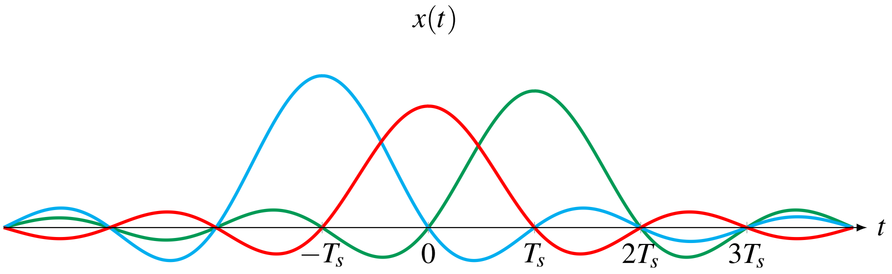
$$\begin{aligned}h(t) = T_s\frac{\sin(\omega_c t)}{\pi t} &= T_s\frac{\sin(\omega_c t)}{\frac{\pi}{\omega_c}\omega_c t}\\ &= \frac{\omega_c T_s}{\pi}\,\mathrm{Sa}(\omega_c t)\\ &= \frac{\omega_c T_s}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c t}{\pi}\right)\end{aligned}$$

and if we use $\omega_c = \text{midpoint}\,(\omega_B, \omega_s - \omega_B) = \frac{\omega_s}{2}$ we have

$$h(t) = \frac{\omega_s T_s}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_s t}{2\pi}\right) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$

and the interpolation formula becomes:

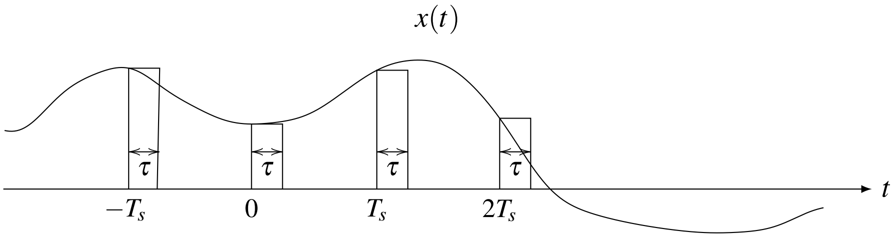
$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,\underbrace{\mathrm{sinc}\left(\frac{t-kT_s}{T_s}\right)}_{=0 \text{ at multiples of } T_s}} \tag{8.9}$$



MATLAB Demo.

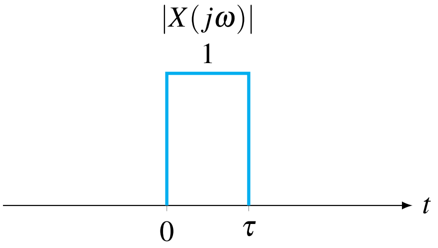
8.3 Non-Ideal Sampling

Reconstruction of $x(t)$ using $\delta_{T_s}(t)$ is not practical (due to the presence of deltas). A more practical reconstruction approach is "sample and hold" or "flat-top sampling" where we construct the signal $x_{SAH}(t)$ as follows:



Let

$$p(t)=\begin{cases}1 & 0<t<\tau\\ 0 & \text{else}\end{cases}$$



Then

$$x_{SAH}(t)=\sum_{k=-\infty}^{+\infty}x(kT_s)\,p(t-kT_s)$$

Can we perfectly recover $x(t)$ from $x_{SAH}(t)$? Let's evaluate $X_{SAH}(j\omega)$. We have:

$$\begin{aligned} x_{SAH}(t) &= \sum_{k=-\infty}^{+\infty}x(kT_s)\,\underbrace{p(t-kT_s)}_{p(t)*\delta(t-kT_s)}\\ \text{(linearity of conv.)} &= p(t)*\underbrace{\sum_{k=-\infty}^{+\infty}x(kT_s)\,\delta(t-kT_s)}_{x_i(t)}\end{aligned}$$

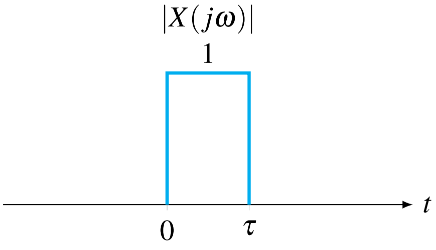
so

$$\begin{aligned} X_{SAH}(j\omega) &= P(j\omega)\cdot X_i(j\omega)\\ &\overset{(8.6)}{=} P(j\omega)\cdot\sum_{n=-\infty}^{+\infty}X\left(j(\omega-n\omega_s)\right)\end{aligned}$$

This is only possible if $P(j\omega)\neq 0$ for all $\omega\in(-\omega_B,\omega_B)$, otherwise $\frac{1}{P(j\omega)}=\infty$!

■ **Example 8.2** If

$p(t)=$

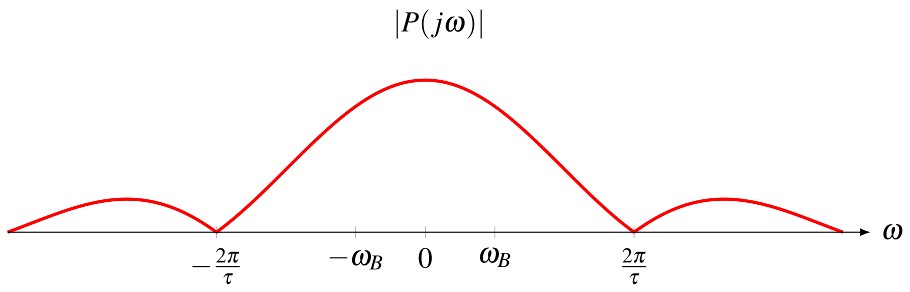


Then

$$P(j\omega)=\frac{\tau}{2}e^{-j\omega\tau}\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

and

$$|P(j\omega)|=\frac{\tau}{2}\left|\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)\right|$$



So, as long as $\frac{2\pi}{\tau} > \omega_B \iff \tau < \frac{2\pi}{\omega_B}$, then $P(j\omega) \neq 0$ for all $\omega \in (-\omega_B, \omega_B)$ and an equalizer exists. In general, as τ becomes smaller, $P(j\omega)$ becomes "**flatter**" over $(-\omega_B, \omega_B)$, and $X_{SAH}(j\omega)$ gets closer to $X_i(j\omega)$. ■

8.4 Natural Sampling

So far we have seen:

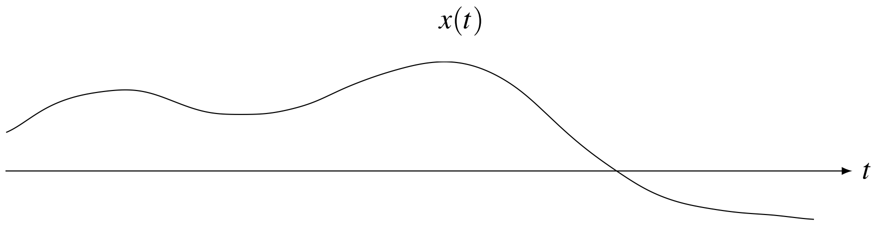
1. Sampling $x(t) \longrightarrow \{x(kT_s)\}_{k=-\infty}^{+\infty}$ with ideal reconstruction, i.e.

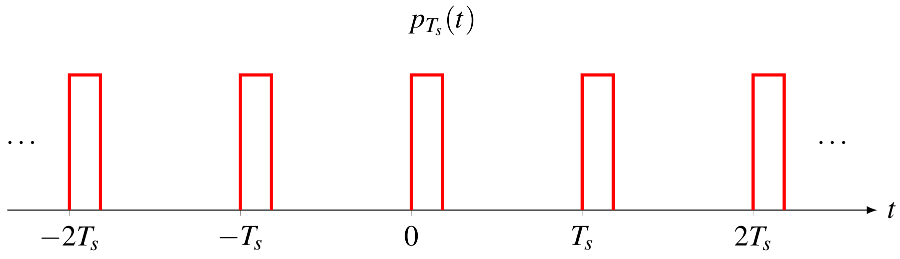
$$x_i(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s) \longrightarrow \boxed{\text{LPF}} \longrightarrow x(t)$$

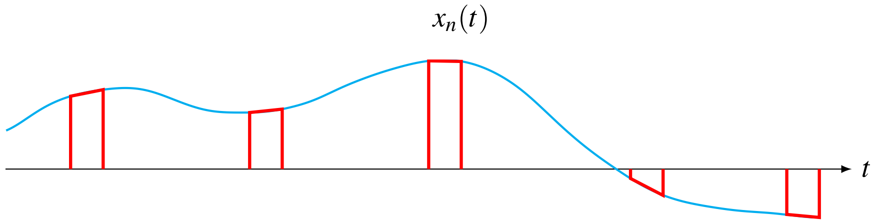
2. Sampling $x(t) \longrightarrow \{x(kT_s)\}_{k=-\infty}^{+\infty}$ with practical/flat-top/sample & hold reconstruction, i.e.

$$x_{SAH}(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,p(t-kT_s) \longrightarrow \boxed{\text{equalizer}} \longrightarrow x(t)$$

There is another type of sampling, called "natural" sampling, where the sampled signal is constructed as follows:



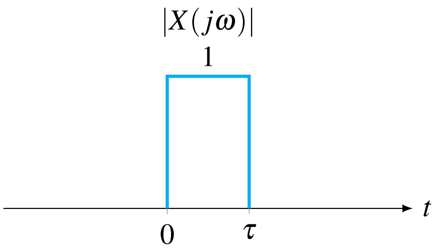




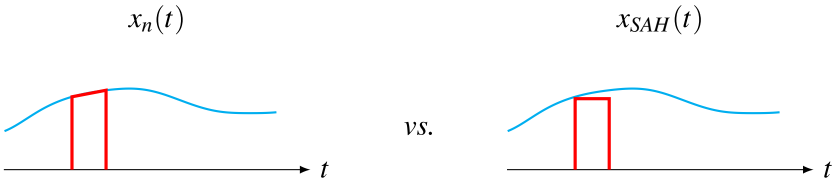
Here $x_n(t) = x(t) \cdot p_{T_s}(t)$ where $p_{T_s}(t)$ is a periodic signal constructed as

$$p_{T_s}(t) = \sum_{k=-\infty}^{+\infty} p(t - kT_s)$$

with $p(t)$ some arbitrary pulse, e.g.



- Observe that $x_N(t)$ **is not** *flat-top* ($\Longleftrightarrow$ SAH) sampling, i.e. $x_n(t) \neq x_{SAH}(t)$ since the ampiltude of $x_n(t)$ "follows" $x(t)$.



- Also note that this is not exactly "sampling" in the sense that $x_n(t)$ retains more information about $x(t)$ than $\{x(kT_s)\}_{k=-\infty}^{+\infty}$.

- Maybe a more appropriate name for generating $x_n(t)$ from $x(t)$ would be "switching."

Question: Can we reconstruct $x(t)$ from $x_n(t)$ and how?

Answer: Let's evaluate $X_n(j\omega)$:

$$\begin{aligned} x_n(t) &= x(t)\cdot p_{T_s}(t) \\ \Longrightarrow\; X_n(j\omega) &= \frac{1}{2\pi}X(j\omega)*P_{T_s}(j\omega) \end{aligned}$$

but

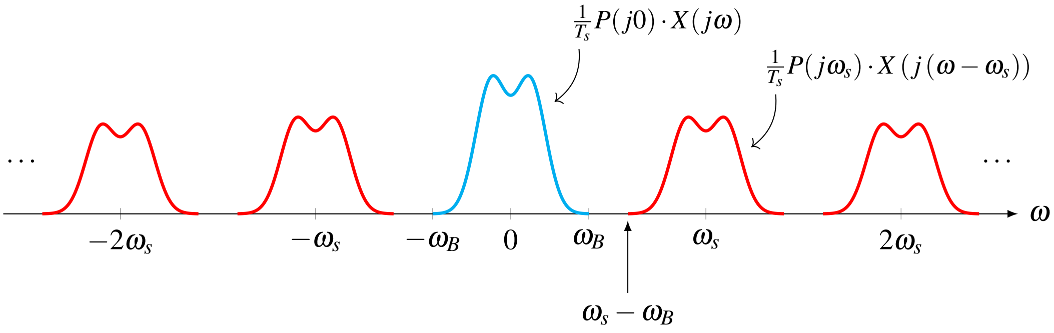
$$\begin{aligned} p_{T_s}(t) &= \sum_{k=-\infty}^{+\infty} p(t-kT_s) = \sum_{k=-\infty}^{+\infty} p(t)*\delta(t-kT_s) \\ &= p(t)*\sum_{k=-\infty}^{+\infty}\delta(t-kT_s) \\ &= p(t)*\delta_{T_s}(t) \end{aligned}$$

so

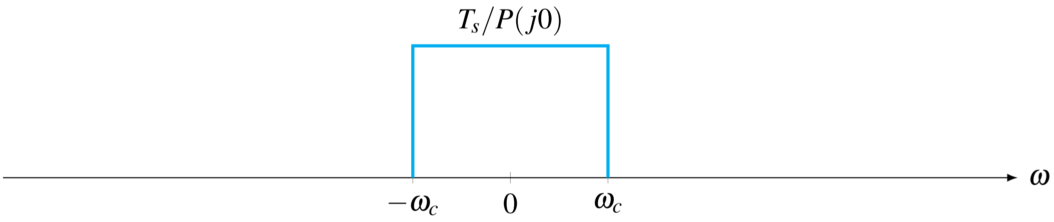
$$\begin{aligned} P_{T_s}(j\omega) &= P(j\omega)\cdot\mathscr{F}\{\delta_{T_s}(t)\} \\ &= P(j\omega)\cdot\frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_s) \\ &= \frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}P(j\omega)\delta(\omega-n\omega_s) \\ \text{(sampling property)}\quad &= \frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}P(jn\omega_s)\delta(\omega-n\omega_s) \end{aligned}$$

and after substitution we get

$$\begin{aligned} X_n(j\omega) &= \frac{1}{2\pi}X(j\omega)*\frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}P(jn\omega_s)\,\delta(\omega-n\omega_s) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{+\infty}P(jn\omega_s)X(j(\omega-n\omega_s)) \end{aligned}$$

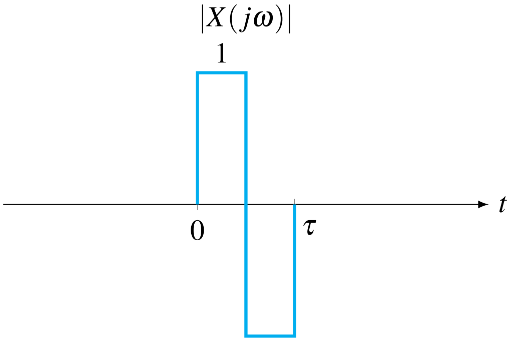


So each replica of $X(j\omega)$ at $n\omega_s$ is scaled by **a constant** $\frac{1}{T_s}P(jn\omega_s)$. Clearly, if $x(t)$ is strictly bandlimited with absolute bandwidth ω_B **and** if the sampling frequency $\omega_s > 2\omega_B$ **and** if $P(j0)\neq 0$ **then** we can recover $x(t)$ from $x_n(t)$ by an ideal low-pass filter:



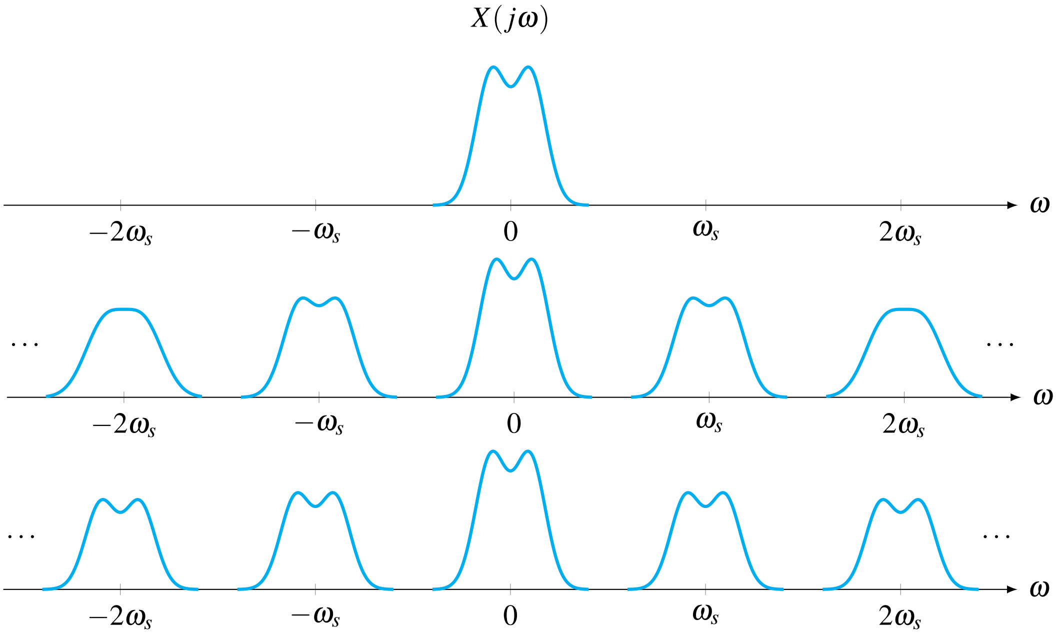
for $\omega_c \in (\omega_B, \omega_s - \omega_B)$.

R $P(j0) \neq 0$ means that the pulse $p(t)$ we are using to construct $p_{T_s}(t)$ has non-zero DC, i.e. $p(t) =$



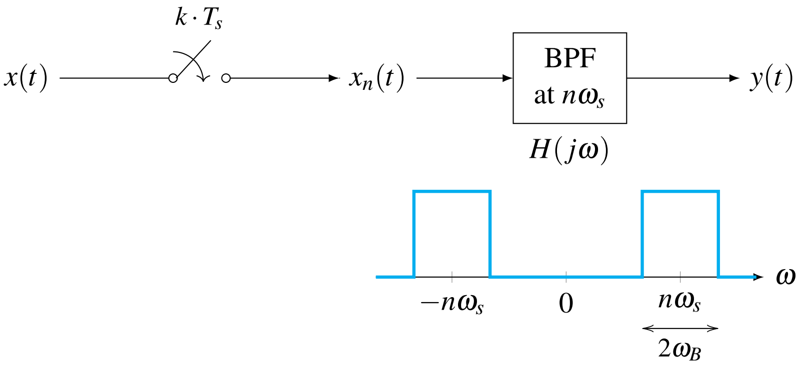
would be an inappropriate pulse to use!

R $x_n(t) \neq x_{SAH}(t) \iff X_n(j\omega) \neq X_{SAH}(j\omega)$.



For $X_{SAH}(j\omega)$, each of the replicas is undistorted by $p(t)$, just scaled.

Note that natural sampling (switching) can be used to modulate a signal:

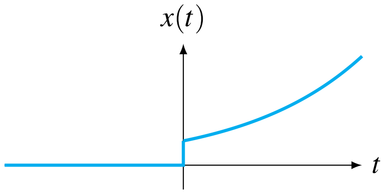


$$
\begin{aligned}
Y(j\omega) &= \frac{1}{T_s}P(jn\omega_s)X(j(\omega - n\omega_s)) \\
&+ \frac{1}{T_s}P(-jn\omega_s)X(j(\omega + n\omega_s))
\end{aligned}
$$

9. Laplace Transform

Motivation 1: Many signals of interest do not have a Fourier Transform, e.g.

$x(t) = e^t u(t)$

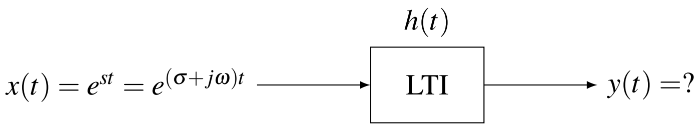


or even $x(t) = u(t)$ (strictly speaking) since they are not absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt = \infty$$

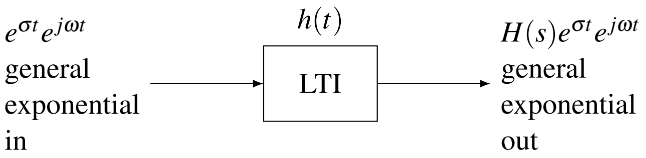
It would be useful to define a new transform such that signals such as the above have such a transform.

Motivation 2: Consider an exponential e^{st} through an LTI system, where $s = \sigma + j\omega$:



$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{H(s),\ s=\sigma+j\omega} \end{aligned}$$

So



where $H(s)$ is the **gain** (assuming it exists). This motivates a new transform.

9.1 Bilateral Laplace Transform

Definition 9.1 — Bilateral Laplace Transform. The ***Bilateral Laplace Transform*** $X(s)$ of signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{9.1}$$

where $s \in \mathbb{C}$, $s = \sigma + j\omega$ where σ is the real part and ω is the imaginary part. The above is defined for all $s \in \mathbb{C}$ for which the integral exists.

R

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t}\right) e^{-j\omega t}dt \\ &= \mathscr{F}\left\{x(t)e^{-\sigma t}\right\} \end{aligned}$$

This exists if $x(t)e^{-\sigma t}$ is absolutely integrable, i.e.

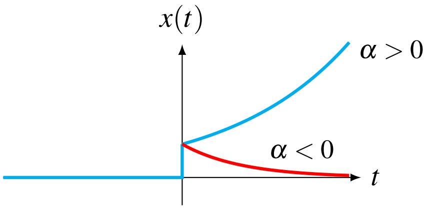
$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$$

The set of complex numbers for which $X(s)$ is well defined is called the ***Region of Convergence*** (ROC) of $X(s)$:

$$ROC = \left\{s \in \mathbb{C} \mid x(t)e^{-\sigma t} = x(t)e^{-\operatorname{Re}(s)} \text{ is abs. integrable}\right\} \tag{9.2}$$

■ **Example 9.1**

$$x(t) = e^{\alpha t}u(t), \quad \alpha \in \mathbb{R}$$



$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{(\alpha-s)t}dt \end{aligned}$$

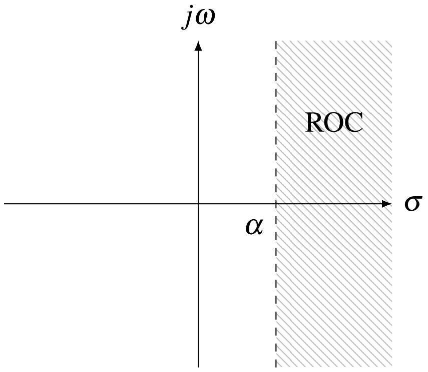
Does this integral exist?

$$\left|e^{(\alpha-s)t}\right| = \left|e^{(\alpha-\sigma-j\omega)t}\right| = e^{(\alpha-\sigma)t}, \quad \alpha < \sigma$$

$$\int_{0}^{\infty} e^{(\alpha-s)t} = \left.\frac{e^{(\alpha-s)t}}{\alpha-s}\right|_{0}^{+\infty} = \frac{0-1}{\alpha-s} = \frac{1}{s-a}$$

so

$$\boxed{X(s) = \frac{1}{s-a}, \quad \operatorname{Re}\{s\} > a} \tag{9.3}$$



■

■ Example 9.2

$$\begin{aligned}
x(t) &= -e^{at}u(-t) \\
X(s) &= \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{at}e^{-st}dt \\
&= -\int_{-\infty}^{0} e^{(a-s)t}dt
\end{aligned}$$

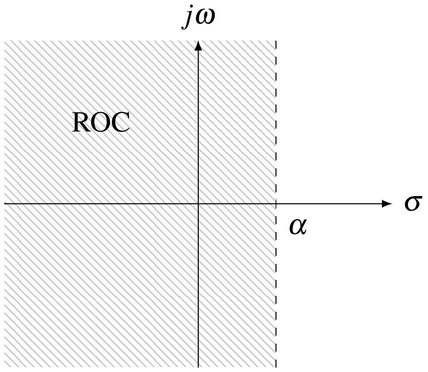
Does the integral exist?

$$\begin{aligned}
&\left|e^{(a-s)t}\right| = \left|e^{(\alpha-\sigma-j\omega)t}\right| = e^{(a-\sigma)t} \\
&a-\sigma > 0 \iff \sigma < a
\end{aligned}$$

$$-\int_{-\infty}^{0} e^{(a-s)t}dt = -\frac{e^{(a-s)t}}{a-s}\bigg|_{-\infty}^{0} = -\frac{1-0}{a-s} = \frac{1}{s-a}$$

So

$$\boxed{X(s) = \frac{1}{s-a}, \ \ \mathrm{Re}\{s\} < a} \tag{9.4}$$



■

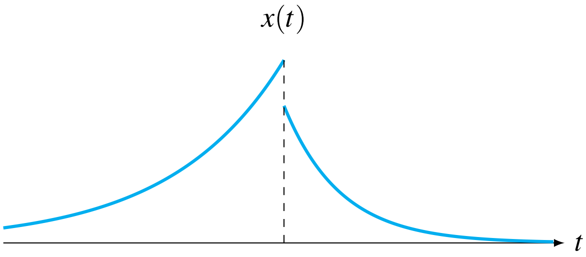
Observe that both examples result in:

$$X(s)=\frac{1}{s-a}$$

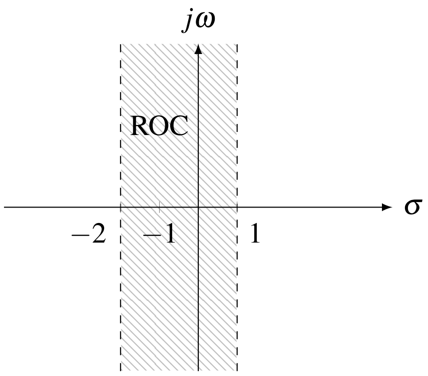
This alone cannot uniquely specify $x(t)$! For a Laplace Transform we need **both** pieces of information, $X(s)$ (the algebraic expression) and the ROC.

■ Example 9.3

$$x(t)=3e^{-2t}u(t)+4e^{t}u(-t)$$



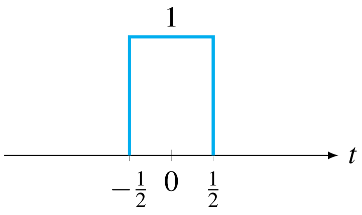
$$\begin{aligned}X(s)&=\int_{-\infty}^{\infty}x(t)e^{-st}dt=\underbrace{\int_{0}^{+\infty}3e^{-2t}e^{-st}dt}_{\text{ex. 1}}+\underbrace{\int_{-\infty}^{0}4e^{t}e^{-st}dt}_{\text{ex. 2}}\\&=3\cdot\underset{\text{Re}\{s\}>-2}{\frac{1}{s+2}}-4\cdot\underset{\text{Re}\{s\}<1}{\frac{1}{s-1}}\\&=\frac{3}{s+2}-\frac{4}{s-1},\quad -2<\text{Re}\{s\}<1\end{aligned}$$



■

■ **Example 9.4**

$x(t) = \text{rect}(t)$

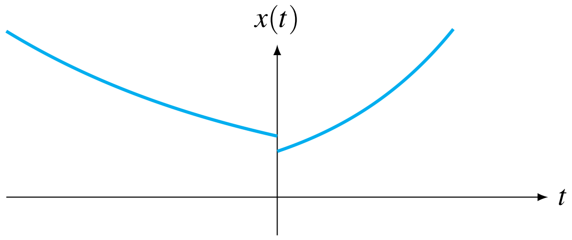


$$X(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-st} dt = \begin{cases} \frac{e^{-st}}{-s}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}, & s \neq 0 \\ 1, & s = 0 \end{cases} = \begin{cases} \frac{e^{-s/2} - e^{+s/2}}{-s}, & s \neq 0 \\ 1, & s = 0 \end{cases} = \begin{cases} \frac{e^{s/2} - e^{-s/2}}{s}, & s \neq 0 \\ 1, & s = 0 \end{cases}$$

Since $1 \cdot e^{-\sigma t} \cdot \text{rect}(t)$ is always absolutely integrable, the integral exists. ■

■ **Example 9.5**

$$x(t) = 3e^{2t}u(t) + 4e^{-t}u(-t)$$



$$X(s) = \underbrace{3 \underbrace{\int_0^{+\infty} e^{2t} e^{-st} dt}_{\frac{1}{s-2},\ \text{Re}\{s\}>2} + 4 \underbrace{\int_{-\infty}^{0} e^{-t} e^{-st}}_{-\frac{1}{s+1},\ \text{Re}\{s\}<-1}}_{ROC=\emptyset}$$

■

So far we have seen 5 different ROC types:

$$ROC \begin{cases} \text{Re}\{s\} > \sigma_{min} \\ \text{Re}\{s\} < \sigma_{max} \\ \sigma_{min} < \text{Re}\{s\} < \sigma_{max} \\ \mathbb{C} \\ \emptyset \end{cases}$$

9.1.1 Observations:

- ROC only depends on $\text{Re}\{s\} = \sigma$:

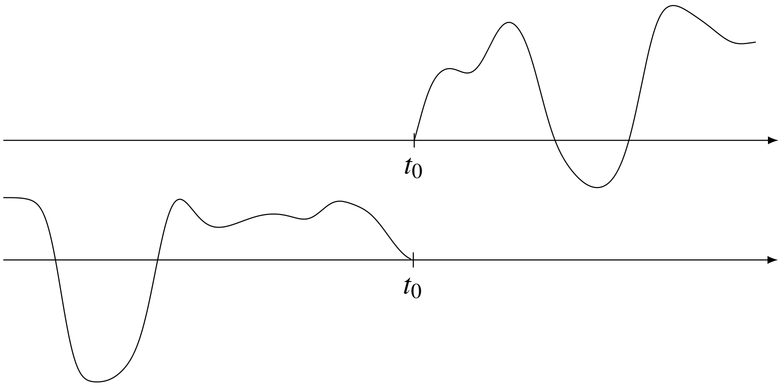
$$\Updownarrow$$

$$s \in \text{ROC} \iff x(t)e^{-\sigma t} \text{ is absolutely integrable} \implies \text{only Re}\{s\} \text{ matters.}$$

- These are the only possible ROC types!

9.2 Right (Left) Handed Signals

Definition 9.2 — Right and Left Handed Signals. A ***right-handed signal*** (RHS) $x(t)$ is a signal for which there exists a t_0 such that $x(t) = 0$ for all $t < t_0$ $(t > t_0)$. A ***left-handed signal*** (LHS) $x(t)$ is a signal for which there exists a t_0 such that $x(t) = 0$ for all $t > t_0$.



ROCs for right-handed signals are:

$$ROC \begin{cases} \mathbb{C} \\ \emptyset \\ \mathrm{Re}\{s\} > \sigma_{min} \end{cases}$$

Proposition 9.1 For a right-handed signal, if $s = \sigma + j\omega \in ROC$ then any $s' = \sigma' + j\omega' \in ROC$ for $\sigma' > \sigma$.

Proof. Assume $s \in \sigma + j\omega \in$ ROC

$$\begin{aligned}
&\Longrightarrow \int_{t_0}^{\infty} |x(t)| e^{-\sigma t} dt < \infty \\
&\Longrightarrow \int_{t_0}^{\infty} |x(t)| e^{-(\sigma+\sigma'-\sigma')t} dt < \infty \\
&\Longrightarrow \int_{t_0}^{\infty} |x(t)| e^{-\sigma' t} e^{\overbrace{(\sigma' - \sigma)}^{>0} t} < \infty \\
&\Longrightarrow \int_{t_0}^{\infty} |x(t)| e^{-\sigma' t} e^{(\sigma'-\sigma)t_0} dt < \infty \\
&\Longrightarrow e^{(\sigma'-\sigma)t_0} \int_{t_0}^{\infty} |x(t)| e^{-\sigma' t} dt < \infty \\
&\Longrightarrow \int_{t_0}^{\infty} |x(t)| e^{-\sigma' t} dt < \infty \\
&\Longrightarrow s' = \sigma' + j\omega' \in ROC
\end{aligned}$$

□

9.3 Inverse Laplace Transform

How do we find $x(t)$ from $X(s)$ and ROC? Pick any $\sigma_0 \in ROC$

Recall:

$$\begin{aligned} X(\sigma_0 + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma_0 + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-\sigma_0 t} e^{-j\omega t} dt \\ &= \mathscr{F}\left\{x(t) e^{-\sigma_0 t}\right\} \end{aligned}$$

$\implies$ using $\mathscr{F}^{-1}$,

$$\begin{aligned} x(t) e^{-\sigma_0 t} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma_0 + j\omega) e^{j\omega t} d\omega \\ \implies x(t) &= \frac{1}{2\pi} e^{-\sigma_0 t} \int_{-\infty}^{\infty} X(\sigma_0 + j\omega) e^{j\omega t} d\omega \\ \implies & \boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma_0 + j\omega) e^{(\sigma_0 + j\omega)t} d\omega} \end{aligned}$$

- $x(t)$ is a weighted sum (actually integral) of general exponentials $e^{\sigma_0 t} e^{j\omega t}$.
- Compare with Fourier Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

  so $x(t)$ is a weighted sum (actually integral) of pure complex exponentials $e^{j\omega t}$ (for all frequencies $\omega \in \mathbb{R}$).

By change of variables you can rewrite the inverse Laplace Transform as follows:
Set $s = \sigma_0 + j\omega$, $ds = jd\omega$:

$$x(t) = \frac{1}{2\pi} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} X(S) e^{st} \frac{ds}{j} = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} X(s) e^{st} ds$$

Let's simplify this expression for real signals $x(t)$.

Proposition 9.2 If $x(t)$ is real, then $X^*(s) = X(s^*)$.

Proof.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ X^*(s) &= \int_{-\infty}^{\infty} x(t)^* e^{-s^* t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-s^* t} dt \\ &= X(s^*) \end{aligned}$$

□

The above proposition is a generalization of Hermitian symmetry.

Now take a real signal $x(t)$.

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\sigma_0+j\omega\right)e^{(\sigma_0+j\omega)td\omega} \\
&= \frac{1}{2\pi}\int_{0}^{+\infty} X\left(\sigma_0+j\omega\right)e^{(\sigma_0+j\omega)t}d\omega \\
&\underbrace{+\frac{1}{2\pi}\int_{-\infty}^{0} X\left(\sigma_0+j\omega\right)e^{(\sigma_0+j\omega)t}d\omega}_{\omega'=-\omega}
\end{aligned}
$$

Using the substitution $\omega' = -\omega$, we can rewrite the second integral as:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{+\infty}^{0} X\left(\sigma_0-j\omega'\right)e^{(\sigma_0-j\omega')t}\left(-d\omega'\right) \\
=&\frac{1}{2\pi}\int_{0}^{+\infty} X\underbrace{\left(\sigma_0-j\omega'\right)}_{(\sigma_0+j\omega)^*}e^{(\sigma_0-j\omega')t}\left(d\omega'\right) \\
=&\frac{1}{2\pi}\int_{0}^{+\infty} X^*\left(\sigma_0+j\omega'\right)e^{(\sigma_0-j\omega')t}d\omega'
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{0}^{+\infty}\left[\underbrace{X\left(\sigma_0+j\omega\right)e^{(\sigma_0+j\omega)t}}_{z}+\underbrace{X^*\left(\sigma_0+j\omega\right)e^{(\sigma_0-j\omega)t}}_{z^*}\right]d\omega \\
&= \frac{1}{2\pi}\int_{0}^{+\infty} 2\operatorname{Re}\{z\}\,d\omega \\
&= \frac{1}{\cancel{2}\pi}\int_{0}^{+\infty} \cancel{2}\operatorname{Re}\left\{\left|X\left(\sigma_0+j\omega\right)\right|e^{\angle X(\sigma_0+j\omega)}e^{(\sigma_0+j\omega)t}\right\}d\omega
\end{aligned}
$$

$$
\boxed{x(t) = \frac{1}{\pi}\int_{0}^{+\infty} \left|X\left(\sigma_0+j\omega\right)\right|e^{\sigma_0 t}\cos\left(\omega t+\angle X\left(\sigma_0+j\omega\right)\right)d\omega}
$$

This is a **weighted** sum (integral) of $\underbrace{\textit{exponentially weighted}}_{e^{\sigma_0 t}}$ $\underbrace{\text{cosines}}_{\cos(\omega t+\angle X(\sigma_0+j\omega))}$.

9.4 Unilateral Laplace Transform

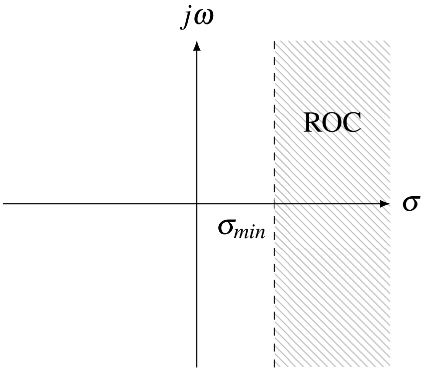
From now on we will only consider RHS (right-handed signals), and in particular, signals with $x(t) = 0$ for all $t < 0$.

Definition 9.3 — Unilateral Laplace Transform. We will define the "***one-sided***" or "***unilateral***" Laplace Transform of $x(t)$ as

$$
X(s) = \int_{0^-}^{+\infty} x(t)e^{-st}dt \tag{9.5}
$$

The lower bound of the integral is 0^- so we can deal with $\delta(t)$, deltas at $t = 0$.

$$
ROC\begin{cases}\mathbb{C}\\ \emptyset\\ \operatorname{Re}\{s\} > \sigma_{min}\ \text{(RHP)}\end{cases}
$$



9.5 Examples of Laplace Transform Pairs

1.

$$\boxed{u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s}} \tag{9.6}$$

Proof.

$$\int_{0^-}^{\infty} u(t)e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}$$

if $\mathrm{Re}\{s\} > 0$.
ROC: $\int_{0^-}^{\infty} |u(t)|e^{-\sigma t}dt < \infty$ if and only if $\sigma > 0$, i.e. $\mathrm{Re}\{s\} > 0$. $\square$

2.

$$\boxed{\delta(t) \overset{\mathscr{L}}{\longleftrightarrow} 1} \tag{9.7}$$

Proof.

$$\int_{0^-}^{\infty} \delta(t)e^{-st}dt = 1, \text{ for all s}$$

ROC: $\int_{0^-}^{\infty} |\delta(t)|e^{-\sigma t}dt < \infty$ for all σ, i.e. **for all s**. $\square$

3.

$$\boxed{e^{-at}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}} \tag{9.8}$$

Proof.

$$\begin{aligned}\int_{0^-}^{\infty} e^{-at}u(t)e^{-st}dt &= \int_0^a e^{-(a+s)t}dt \\ &= \frac{1}{s+a}e^{-(a+s)t}\Big|_0^a \\ &= \frac{1}{s+a}, \ \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}\end{aligned}$$

ROC: $\int_{0^-}^{\infty} \left|e^{-(a+s)t}\right| dt = \int_{0^-}^{\infty} e^{-[\mathrm{Re}\{a\}+\mathrm{Re}\{s\}]t}dt$.
$\therefore \int_{0^-}^{\infty} \left|e^{-(a+s)t}\right| dt < \infty$ if an only if $\mathrm{Re}\{a\} + \mathrm{Re}\{s\} > 0 \implies \mathrm{Re}\{\mathbf{s}\} > -\mathrm{Re}\{\mathbf{a}\}$ $\square$

4.

$$\boxed{\cos(\omega t)u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{s}{s^2+\omega^2}} \tag{9.9}$$

(ω real-valued).

Proof.

$$\begin{aligned}
\int_{0^-}^{\infty} \cos(\omega t)u(t)e^{-st}dt &= \int_0^{\infty} \frac{1}{2}e^{j\omega t}e^{-st}dt + \int_0^{\infty} \frac{1}{2}e^{-j\omega t}e^{-st}dt \\
&= \frac{1}{2}\frac{-1}{s-j\omega}e^{-(s-j\omega)t}\Big|_0^{\infty} + \frac{1}{2}\frac{1}{s+j\omega}e^{-(s+j\omega)t}\Big|_0^{\infty} \\
&= \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} \text{ if } \mathrm{Re}\{s\} > 0 \\
&= \frac{s}{s^2+\omega^2}, \text{ if } \mathrm{Re}\{s\} > 0
\end{aligned}$$

ROC: $\int_0^{\infty} \left|e^{\pm j\omega t} \cdot e^{-st}\right| dt = \int_0^{\infty} e^{-\sigma t} dt < \infty$ if and only if $\sigma > 0$, i.e. $\mathrm{Re}\{s\} > 0$. ☐

R The ROC's of each of these four Laplace Transforms are RHP's, or in the case of (2) the entire $s-$plane.

Signal	Transform	ROC
1. $u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
2. $u(t) - u(t-a)$	$\frac{1-e^{-as}}{s}$	For all s
3. $\delta(t)$	1	For all s
4. $\delta(t-a)$	e^{-as}	For all s
5. $t^n u(t)$	$\frac{n!}{s^{n+1}}, n = 1, 2, \ldots$	$\mathrm{Re}\{s\} > 0$
6. $e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$
7. $t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
8. $\cos\omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
9. $\sin\omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
10. $\cos^2\omega_0 t u(t)$	$\frac{s^2+2\omega_0^2}{s\left(s^2+4\omega_0^2\right)}$	$\mathrm{Re}\{s\} > 0$
11. $\sin^2\omega_0 t u(t)$	$\frac{2\omega_0^2}{s\left(s^2+4\omega_0^2\right)}$	$\mathrm{Re}\{s\} > 0$
12. $e^{-at}\cos\omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
13. $e^{-at}\sin\omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
14. $t\cos\omega_0 t u(t)$	$\frac{s^2-\omega_0^2}{\left(s^2+\omega_0^2\right)^2}$	$\mathrm{Re}\{s\} > 0$
15. $t\sin\omega_0 t u(t)$	$\frac{2\omega_0 s}{\left(s^2+\omega_0^2\right)^2}$	$\mathrm{Re}\{s\} > 0$

Table 9.1: Some Selected Unilateral Laplace Transform Pairs

In the above table, α and ω_0 are real-valued quantities.

9.6 Laplace Transform Properties

9.6.1 Linearity

If

$$x_1(t) \xleftrightarrow{\mathscr{L}} X_1(s),\ \ ROC\ R_1$$

and

$$x_2(t) \xleftrightarrow{\mathscr{L}} X_2(s),\ \ ROC\ R_2$$

then

$$\boxed{ax_1(t) + bx_2(t) \xleftrightarrow{\mathscr{L}} aX_1(s) + bX_2(s)} \tag{9.10}$$

and $R_1 \cap R_2 \subseteq R^+$, where R^+ is the ROC[1].

■ Example 9.6

$$\begin{aligned} x(t) &= Au(t) + Be^{-bt}u(t),\ \ b \text{ real-valued} \\ u(t) &\xleftrightarrow{\mathscr{L}} \frac{1}{s},\ \text{Re}\{s\} > 0 \\ e^{-bt}u(t) &\xleftrightarrow{\mathscr{L}} \frac{1}{s+b},\ \text{Re}\{s\} > -b \\ \therefore\ x(t) &\xleftrightarrow{\mathscr{L}} \frac{A}{s} + \frac{B}{s+b},\ \text{Re}\{s\} > \max(0, -b) \end{aligned}$$

■

9.6.2 Time Shifting

If

$$x(t) \xleftrightarrow{\mathscr{L}} X(s),\ \ ROC\ R$$

then for any **positive** real number t_0,

$$\boxed{x(t - t_0) \xleftrightarrow{\mathscr{L}} e^{-t_0 s}X(s)},\ \ ROC\ R \tag{9.11}$$

Proof.

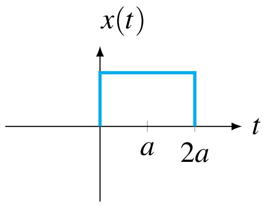
$$\begin{aligned} \int_{0^-}^{\infty} x(t - t_0)e^{-st}dt &= \int_{0^- - t_0}^{\infty} x(\tau)e^{-s(\tau+t_0)d\tau} \text{ (let } \tau \equiv t - t_0) \\ &= e^{-st_0}\int_{0^-}^{\infty} x(\tau)e^{-s\tau}d\tau \text{ (since } x(\tau) = 0 \text{ for } \tau < 0) \\ &= e^{-st_0}X(s) \end{aligned}$$

□

---

[1] The ROC of $ax_1(t) + bx_2(t)$ can in some cases be larger than $R_1 \cap R_2$. For a trivial example, let $x_1(t) = \delta(t) + e^{-t}u(t)$ and $x_2(t) = -e^{-t}u(t)$. Then $X_1(s) = 1 + \frac{1}{s+1}$, $\text{Re}\{s\} > -1$ and $X_2(s) = -\frac{1}{s+1}$, $\text{Re}\{s\} > -1$ and $Y(s) = 1$, for all s, where $y(t) = x_1(t) + x_2(t)$.

■ Example 9.7



$$x(t) = \text{rect}\left(\frac{t-a}{2a}\right), \; a > 0$$

$$\therefore \; x(t) = u(t) - u(t-2a) = \frac{1}{s} - e^{-2as} \cdot \frac{1}{s}$$

$$\therefore \; X(s) = \frac{1}{s}\left[1 - e^{-2as}\right], \text{ ROC is entire } s\text{−plane}$$

■

9.6.3 Modulation

If

$$x(t) \xleftrightarrow{\mathscr{L}} X(s), \;\; ROC: \; \text{Re}\{s\} > \sigma_0$$

then

$$\boxed{e^{s_0 t}x(t) \xleftrightarrow{\mathscr{L}} X(s - s_0)}, \;\; ROC: \text{Re}\{s\} > \sigma_0 + \text{Re}\{s_0\} \tag{9.12}$$

Proof.

$$\begin{aligned} \int_{0^-}^{\infty} e^{s_0 t} x(t) e^{-st} dt &= \int_{0^-}^{\infty} x(t) e^{-(s-s_0)t} dt \\ &= X(s - s_0), \; \text{Re}\{s - s_0\} > \sigma_0 \end{aligned}$$

□

■ Example 9.8

$$\cos \omega_0 t u(t) \xleftrightarrow{\mathscr{L}} \frac{s}{s^2 + \omega_0^2}, \; ROC: \text{Re}\{s\} > 0$$

$$\therefore \; e^{-at} \cos \omega_0 t u(t) \xleftrightarrow{\mathscr{L}} \frac{s+a}{(s+a)^2 + \omega_0^2}, \; ROC: \text{Re}\{s\} > -a$$

(where a is real-valued). ■

9.6.4 Time Scaling

If

$$x(t) \xleftrightarrow{\mathscr{L}} X(s), \;\; ROC: \; \text{Re}\{s\} > \sigma_0$$

then for any **positive** real number α,

$$\boxed{x(\alpha t) \xleftrightarrow{\mathscr{L}} \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right)}, \;\; \text{Re}\{s\} > \alpha\sigma_0 \tag{9.13}$$

Proof.

$$\begin{aligned} \int_{0^-}^{\infty} x(\alpha t) e^{-st} dt &= \frac{1}{\alpha} \int_{0^-}^{\infty} x(\tau) e^{-\frac{s}{\alpha}\tau} d\tau \text{ (let } \tau = \alpha t) \\ &= \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right), \; ROC: \text{Re}\left\{\frac{s}{a}\right\} > \sigma_0 \end{aligned}$$

□

9.6.5 Differentiation

If

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s), \quad ROC: \ \mathrm{Re}\{s\} > \sigma_0$$

then

$$\boxed{\frac{dx(t)}{dt} \overset{\mathscr{L}}{\longleftrightarrow} sX(s) - x\left(0^-\right)}, \quad ROC: \ \mathrm{Re}\{s\} > \sigma_1, \ \sigma_1 \leq \sigma_0 \tag{9.14}$$

The ROC is at least as large as the initial ROC.

Proof.

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

Let $u \equiv e^{-st}$, $dv = dx(t) \therefore du = -se^{-st}dt$, $v = x(t)$ (integration by parts).

$$\begin{aligned}
\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt &= \int_{0^-}^{\infty} udv \\
&= uv|_{0^-}^{\infty} - \int_{0^-}^{\infty} vdu \\
&= x(t)e^{-st}|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}dt \\
&= -x\left(0^-\right) + sX(s)
\end{aligned}$$

Note that if s is in the ROC of $X(s)$, then $\int_{0^-}^{\infty} |x(t)|e^{-(\mathrm{Re}\{s\})t}dt < \infty$, and $\therefore \lim_{t\to\infty} x(t)e^{-st} = 0$. ☐

This result can easily be extended to higher order derivatives. Let $x^{(n)}(t) \equiv \frac{d^n x(t)}{dt^n}$ and $\mathscr{L}\{\cdot\}$ denotes Laplace Transform. Then

$$\begin{aligned}
\mathscr{L}\left\{x^{(2)}(t)\right\} &= s\mathscr{L}\left\{x^{(1)}(t)\right\} - x^{(1)}(0) \\
&= s\left[sX(s) - x\left(0^-\right)\right] - x^{(1)}(0) \\
&= s^2X(s) - sx\left(0^-\right) - x^{(1)}(0) \qquad (9.15)
\end{aligned}$$

Similarly,

$$\mathscr{L}\left\{x^{(3)}(t)\right\} = s^3X(s) - s^2x\left(0^-\right) - sx^{(1)}(0) - x^{(2)}(0) \tag{9.16}$$

$$\mathscr{L}\left\{x^{(4)}(t)\right\} = s^4X(s) - s^3x\left(0^-\right) - s^2x^{(1)}(0) - sx^{(2)}(0) - x^{(3)}(0) \tag{9.17}$$

$$\cdots$$

$$\boxed{\mathscr{L}\left\{x^{(n)}(t)\right\} = s^nX(s) - \sum_{k=0}^{n-1} s^k x^{(n-1-k)}\left(0^-\right)} \tag{9.18}$$

where $x^{(0)} \equiv x$.

The differentiation property makes the Laplace Transform useful in solving ordinary, linear, constant coefficient differential equations. Specifically, we can use the Laplace Transform to convert such a differential equation into an algebraic equation.

■ Example 9.9

$$y''(t)+3y'(t)+2y(t)=0,\ y\left(0^-\right)=3\ \&\ y'\left(0^-\right)=1$$

Taking the Laplace Transform of both sides,

$$\begin{aligned}\therefore\ \left[s^2Y(s)-sy\left(0^-\right)-y'\left(0^-\right)\right]+3\left[sY(s)-y\left(0^-\right)\right]+2Y(s)=0\\ \therefore\ Y(s)\left[s^2+3s+2\right]&=3s+10\\ Y(s)&=\frac{3s+10}{s^2+3s+2}\\ &=\frac{7}{s+1}-\frac{4}{s+2}\\ \therefore\ y(t)&=\mathscr{L}^{-1}\left\{\frac{7}{s+1}-\frac{4}{s+2}\right\}\\ &=7e^{-t}u(t)-4e^{-2t}u(t)\end{aligned}$$

■

■ Example 9.10

$$\begin{aligned}y'(t)+2y(t)=u(t),\ y\left(0^-\right)=2\\ \therefore\ sY(s)-y\left(0^-\right)+2Y(s)&=\frac{1}{s}\\ \therefore\ Y(s)&=\frac{2s+1}{s(s+2)}=\frac{1/2}{s}+\frac{3/2}{s+2}\\ \therefore\ y(t)&=\frac{1}{2}u(t)+\frac{3}{2}e^{-2t}u(t)\end{aligned}$$

■

9.6.6 Integration

If

$$x(t)\overset{\mathscr{L}}{\longleftrightarrow}X(s),\ \ ROC:\ \operatorname{Re}\{s\}>\sigma_0$$

then

$$\boxed{\int_{0^-}^{t}x(\tau)d\tau\overset{\mathscr{L}}{\longleftrightarrow}\frac{X(s)}{s}},\ \ ROC:\ \operatorname{Re}\{s\}>\max(0,\sigma_0) \tag{9.19}$$

Proof. Let

$$\begin{aligned}y(t)\equiv\int_{0^-}^{t}x(\tau)d\tau\implies\frac{dy(t)}{dt}=x(t)\ \&\ y\left(0^-\right)=0\\ \therefore\ Y(s)=\int_{0^-}^{\infty}y(t)e^{-st}dt\end{aligned}$$

Now integrate by parts by letting $u=y(t)$, $dv=e^{-st}dt$.

$$\begin{aligned}Y(s)&=\int_{0^-}^{\infty}y(t)e^{-st}dt\\ &=y(t)\frac{e^{-st}}{-s}\Big|_{0^-}^{\infty}-\int_{0^-}^{\infty}\frac{e^{-st}}{-s}dy(t)\\ &=-\frac{1}{s}\lim_{t\to\infty}\left[y(t)e^{-st}\right]+\frac{y\left(0^-\right)}{s}+\frac{1}{s}\int_{0^-}^{\infty}x(t)e^{-st}dt\\ &=\frac{1}{s}X(s)-\frac{1}{s}\lim_{t\to\infty}\left[y(t)e^{-st}\right]\end{aligned}$$

Now if $s \in ROC$ of $Y(s)$, then $\int_{0^-}^{\infty} |y(t)e^{-st}|\, dt < \infty$ and thus $\lim_{t\to\infty} |y(t)e^{-st}| = 0$. Hence,

$$Y(s) = \frac{1}{s}X(s)$$

□

Thus integration in the time domain is equivalent to division by s in the $s-$domain.

■ Example 9.11

$$u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s} \quad ROC: \ \mathrm{Re}\{s\} > 0$$

$$\therefore \ tu(t) = \int_{0^-}^{t} u(\tau)d\tau \xleftrightarrow{\mathscr{L}} \frac{1}{s} \cdot \left(\frac{1}{s}\right) = \frac{1}{s^2}, \quad ROC: \ \mathrm{Re}\{s\} > 0$$

■

9.6.7 Multiplication by t^n

If

$$x(t) \xleftrightarrow{\mathscr{L}} X(s), \quad ROC: \ \mathrm{Re}\{s\} > \sigma_0$$

then

$$\boxed{t^n x(t) \xleftrightarrow{\mathscr{L}} (-1)^n \frac{d^n X(s)}{ds^n}}, \quad ROC: \ \mathrm{Re}\{s\} > \sigma_0 \tag{9.20}$$

Proof.

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} x(t)e^{-st}dt \\ \therefore \ \frac{d^n X(s)}{ds^n} &= \int_{0^-}^{\infty} x(t)\frac{d^n e^{-st}}{ds^n}dt \\ &= \int_{0^-}^{\infty} (-t)^n x(t)e^{-st}dt \\ &= (-1)^n \int_{0^-}^{\infty} [t^n x(t)]\, e^{-st}dt \end{aligned}$$

Note: repeated powers of t will not affect the ROC.

$$\therefore \ t^n x(t) \xleftrightarrow{\mathscr{L}} (-1)^n \frac{d^n X(s)}{ds^n}$$

□

■ Example 9.12

$$x(t) = e^{-at}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s+a}, \quad ROC: \ \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

$$\therefore \ t^n e^{-at}u(t) \xleftrightarrow{\mathscr{L}} \frac{n!}{(s+a)^{n+1}}, \quad ROC: \ \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

$$\therefore \ \boxed{\frac{t^{n-1}}{(n-1)!}e^{-at} \xleftrightarrow{\mathscr{L}} \frac{1}{(s+a)^n}}, \quad ROC: \ \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

■

9.6.8 Convolution

If

$$x_1(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s),\ \ ROC:\ \operatorname{Re}\{s\} > \sigma_1$$
$$x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_2(s),\ \ ROC:\ \operatorname{Re}\{s\} > \sigma_2$$

then

$$\boxed{x_1(t) * x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s)X_2(s)} \tag{9.21}$$

ROC: Is at least as large as $\operatorname{Re}\{s\} > max(\sigma_1, \sigma_2)$.

Proof.

$$\begin{aligned}
x_1(t) * x_2(t) &= \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau d\tau) \\
&= \int_{0^-}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \text{ (since } x_1(t)=0 \text{ for } t<0) \\
\therefore\ \mathscr{L}\{x_1(t) * x_2(t)\} &= \int_{0^-}^{\infty}\left[\int_{0^-}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-st}dt \\
\text{(switch order of integration)}\ &= \int_{0^-}^{\infty} x_1(\tau)\left[\int_{0^-}^{\infty} x_2(t-\tau)e^{-st}dt\right]d\tau
\end{aligned}$$

Now do a change of variables $\eta = t-\tau$:

$$\begin{aligned}
\int_{0^-}^{\infty} x_1(\tau)\left[\int_{0^-}^{\infty} x_2(t-\tau)e^{-st}dt\right]d\tau &= \int_{0^-}^{\infty} x_1(\tau)\left[\int_{0^- -\tau}^{\infty} x_2(\eta)e^{-s(\eta+\tau)}d\eta\right]d\tau \\
&= \int_{0^-}^{\infty} x_1(\tau)\left[\int_{0^-}^{\infty} x_2(\eta)e^{-s\eta}d\tau\right]e^{-s\tau}d\tau \\
&= \int_{0^-}^{\infty} x_1(\tau)e^{-s\tau}d\tau \cdot \int_{0^-}^{\infty} x_2(\eta)e^{-s\eta}d\eta \\
&= X_1(s)X_2(s),\ \ ROC: \text{at least as large as } \operatorname{Re}\{s\} > \max(\sigma_1, \sigma_2)
\end{aligned}$$

□

■ Example 9.13

$$\begin{aligned}
x_1(t) = e^{-t}u(t) &\overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+1},\ \ ROC:\ \operatorname{Re}\{s\} > -1 \\
x_2(t) = e^{-2t}u(t) &\overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+2},\ \ ROC:\ \operatorname{Re}\{s\} > -2 \\
x_1(t) * x_2(t) &= \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)e^{-2\tau}u(\tau)d\tau \\
&= e^{-t}\int_{0}^{t} e^{-\tau}d\tau u(t) \\
&= \left(e^{-t}-e^{-2t}\right)u(t) \\
\mathscr{L}\left\{\left[e^{-t}-e^{-2t}\right]u(t)\right\} &= \int_{0^-}^{\infty}\left(e^{-t}-e^{-2t}\right)e^{-st}dt \\
&= \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1_(s+2))},\ \ ROC:\ \operatorname{Re}\{s\} > -1 \\
&= X_1(s)X_2(s)
\end{aligned}$$

■

9.6.9 Final Value Theorem

If

$$\lim_{t\to\infty} x(t)$$

exists, then

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)} \tag{9.22}$$

Proof.

$$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \left[sX(s) - x\left(0^-\right)\right]$$

$$\therefore \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \lim_{s\to 0} [sX(s)] - x\left(0^-\right)$$

$$\therefore \int_{0^-}^{\infty} \frac{dx(t)}{dt} \lim_{s\to 0} \left(e^{-st}\right) dt = \lim_{s\to 0} [sX(s)] - x\left(0^-\right)$$

$$\therefore \int_{0^-}^{\infty} \frac{dx(t)}{dt} = \lim_{s\to 0} [sX(s)] - x\left(0^-\right)$$

$$\therefore \lim_{t\to\infty} x(t) - x\left(0^-\right) = \lim_{s\to 0} [sX(s)] - x\left(0^-\right)$$

if $\lim_{t\to\infty} x(t)$ exists. □

■ Example 9.14

$$x(t) = 3u(t) + e^{-2t}u(t)$$

then

$$X(s) = \frac{3}{s} + \frac{1}{s+2}, \ \mathrm{Re}\{s\} > 0$$

$\lim_{t\to\infty} x(t)$ exits and is equal to

$$\lim_{t\to\infty} \left[3u(t) + e^{-2t}u(t)\right] = 3$$

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left[\frac{3}{s} + \frac{1}{s+2}\right] = 3$$

$$\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

■

■ Example 9.15

$$x(t) = \cos(\omega_0 t)u(t)$$

then

$$X(s) = \frac{s}{s^2 + \omega_0^2}, \ \mathrm{Re}\{s\} > 0$$

$$\therefore \lim_{s\to 0} sX(s) = \lim_{s\to 0} s \cdot \frac{s}{s^2 + \omega_0^2} = 0 \quad \neq \underbrace{\lim_{t\to\infty} x(t)}_{\substack{= \lim_{t\to\infty}\cos(\omega_0 t) \\ \text{which does} \\ \text{not exist}}}$$

■

9.7 Poles and Zeros of Laplace Transform

Let $X(s)$ be the Laplace Transform of $x(t)$.

Definition 9.4 — Zeros. s_0 is called a "***zero***" of $X(s)$ if

$$X(s_0) = 0 \tag{9.23}$$

Definition 9.5 — Poles. s_0 is called a "***pole***" of $X(s)$ if

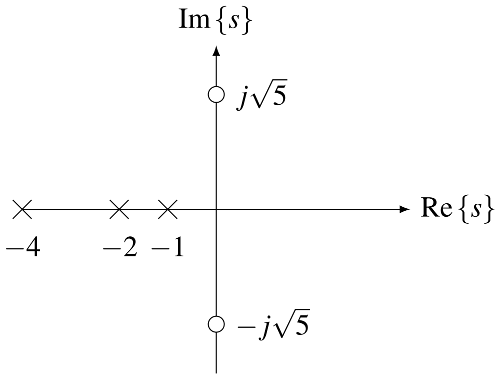
$$|X(s_0)| = \infty \tag{9.24}$$

■ **Example 9.16**

$$X(s) = \frac{s^2+5}{s^3+7s^2+14s+8} = \frac{\left(s+j\sqrt{5}\right)\left(s-j\sqrt{5}\right)}{(s+1)(s+2)(s+4)}$$

$\pm j\sqrt{5}$ are zeros of $X(s)$
-1, -2, -4 are the poles of $X(s)$

It is always instructive to sketch poles/zeros on the $s-$plane. E.g.



The circles denote zeros and the crosses denote poles. ■

Proposition 9.3 If $x(t)$ is real, then poles and zeros of $X(s)$ appear in complex conjugate pairs. I.e., if

$$X(s_0) = 0 \implies X(s_0^*) = 0$$

and if

$$|X(s_0)| = \infty \implies |X(s_0^*)| = \infty$$

Proof. Recall that for real $x(t)$, $X^*(s) = X(s^*)$, so if $X(s_0) = 0 \implies X^*(s_0) = 0 \implies X(s_0^*) = 0$, and similarly for poles. ☐

9.8 Rational Laplace Transforms

Definition 9.6 — Rational Laplace Transform $X(s)$. We say that $X(s)$ is ***rational*** if it can be written as

$$X(s) = \frac{N(s)}{D(s)} \tag{9.25}$$

$$= \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{l=0}^{n} a_l s^l} \tag{9.26}$$

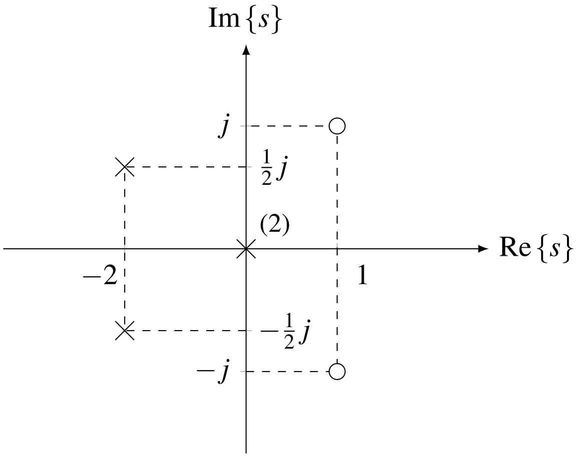
where $N(s)$ and $D(s)$ are polynomials of s. If $\deg(N) = m < \deg(D) = n$ (note that this is a *strict inequality*), we say this rational form is in "**proper**" form. Else, it is in "**improper**" form.

Clearly, poles of $X(s)$ are the roots of $D(s)$ and zeros are the roots of $N(s)$.

■ Example 9.17

$$X(s) = \frac{s^2 - 2s + 2}{s^4 + 4s^3 + 4.25s^2} = \frac{[s-(1+j)]\,[s-(1-j)]}{s^2\left[s-\left(-2+\frac{1}{2}j\right)\right]\left[s-\left(-2-\frac{1}{2}j\right)\right]}$$

The zeros are $1+j$, $1-j$, and the poles are 0 (twice), $-2+\frac{1}{2}j$, $-2-\frac{1}{2}j$.



■

9.8.1 Partial Fraction Expansion (PFE) and Inverse Laplace Transform

Finding the inverse Laplace Transform of a rational $X(s)$ is a simple procedure (algorithm) that requires **Partial Fraction Expansion** (PFE) of $X(s) = \frac{N(s)}{D(s)}$.

Definition 9.7 — Partial Fraction Expansion (PFE). Assume $X(s) = \frac{N(s)}{D(s)}$ is given (proper), and assume the distinct poles are:

$$\begin{array}{rcccccc} & s_1, & s_2, & s_3, & \ldots, & s_p \\ \text{with multiplicity} & \downarrow & \downarrow & \downarrow & & \downarrow \\ & k_1 & k_2 & & & k_p \end{array}$$

Then, the ***Partial Fraction Expansion*** (PFE) of $X(s)$ is

$$\begin{aligned}X(s) = &\frac{A_1^{(1)}}{s-s_1} + \frac{A_2^{(1)}}{(s-s_1)^2} + \dots + \frac{A_{k_1}^{(1)}}{(s-s_1)} \\ &+ \frac{A_1^{(2)}}{s-s_2} + \frac{A_2^{(2)}}{(s-s_2)^2} + \dots + \frac{A_{k_2}^{(2)}}{(s-s_2)^{k_2}} \\ &+ \dots \\ &+ \frac{A_1^{(p)}}{s-s_p} + \frac{A_2^{(p)}}{(s-s_p)^2} + \dots + \frac{A_{k_p}^{(p)}}{(s-s_p)^{k_p}}\end{aligned} \tag{9.27}$$

R Each distinct pole s_l contributes k_l terms in the PFE.

- We will see shortly how to compute the coefficients $A_i^{(j)}$ in the PFE.
- For now assume that they are computed.

How can we find $x(t)$ from $X(s)$? Recall:

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}$$

ROC: $\text{Re}\{s\} > -\text{Re}\{a\}$

■ Example 9.18

$$X(s) = \frac{3s^2+2s+5}{(s+5)^2\cdot(s+3)^3\cdot(s+2)}$$

Poles:	-5	-3	-2
Mult:	2	3	1

$$\begin{aligned}X(s) = &\frac{A_1^{(1)}}{s+5} + \frac{A_2^{(1)}}{(s+5)^2} \\ &+ \frac{A_1^{(2)}}{s+3} + \frac{A_2^{(2)}}{(s+3)^2} + \frac{A_3^{(2)}}{(s+3)^{(3)}} \\ &+ \frac{A_1^{(3)}}{s+2}\end{aligned}$$

$$\begin{aligned}x(t) = &A_1^{(1)}e^{-5t}u(t) + A_2^{(1)}te^{-5t}u(t) \\ &+ A_1^{(2)}e^{-3t}u(t) + A_2^{(2)}te^{-3t}u(t) + A_3^{(2)}\frac{t^2}{2}e^{-3t}u(t) \\ &+ A_1^{(3)}e^{-2t}u(t)\end{aligned}$$

■

■ Example 9.19

$$X(s) = \frac{s^2-2s+2}{s^4+4s^3+5s^2} = \frac{s^2-2s+2}{s^2\,[s-(-2+j)]\,[s-(-2-j)]}$$

Poles:	0	$-2+j$	$-2-j$
Mult:	2	1	1

Note that the poles $-2+j$ and $-2-j$ form a complex conjugate pair.

$$\begin{aligned} X(s) &= \frac{A_1^{(1)}}{s} + \frac{A_2^{(1)}}{s^2} + \frac{A_1^{(2)}}{s-(-2+j)} + \frac{A_1^{(3)}}{s-(-2-j)} \\ x(t) &= A_1^{(1)}u(t) + A_2^{(1)}tu(t) \\ &\quad + A_1^{(2)}e^{(-2+j)t}u(t) + A_1^{(3)}e^{(-2-j)t}u(t) \end{aligned}$$

Since $x(t)$ is real, it has to be that $A_1^{(3)} = A_1^{(2)*}$. So combining the terms we get:

$$\begin{aligned} x(t) &= A_1^{(1)}u(t) + A_2^{(1)}tu(t) \\ &\quad + 2\,\mathrm{Re}\left\{A_1^{(2)}e^{(-2+j)t}u(t)\right\} \\ &= A_1^{(1)}u(t) + A_1^{(2)}tu(t) + 2\left|A_1^{(2)}\right|e^{-2t}\cos\left(t + \angle A_1^{(2)}\right)u(t) \end{aligned}$$

This is an exponentially decaying cosine with frequency $\mathrm{Im}\{-2+j\} = 1$ and growth rate $\mathrm{Re}\{-2+j\} = -2$. ■

9.8.2 Generalization

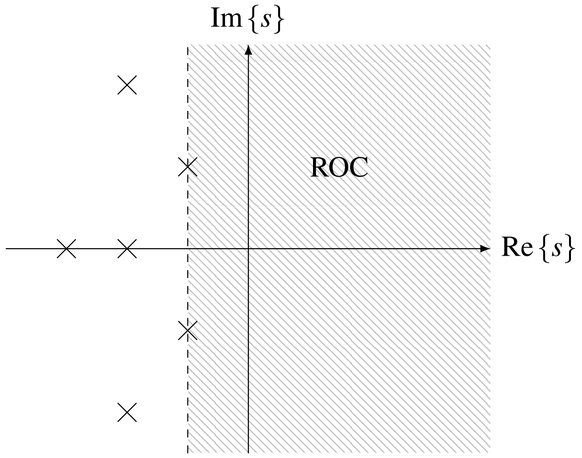
- Every real pole $s_l = \sigma_l + j0$ results in the following time-domain signal:

$$\underbrace{\left[A_1^{(l)} + A_2^{(l)}t + A_3^{(l)}\frac{t^2}{2} + \cdots + A_{k_e}^{(l)}\frac{t^{k_l-1}}{(k_l-1)!}\right]}_{\text{polynomial behavior}} \quad \underset{\substack{\uparrow \\ \text{exponential} \\ \text{behavior}}}{e^{+\sigma_l t}u(t)} \tag{9.28}$$

- Every pair of complex conjugate poles $s_l = \sigma_l + j\omega_l$, $s_k = s_l^* = \sigma_l - j\omega_l$ results in

$$\begin{aligned} &\left[A_1^{(l)} + A_2^{(l)}t + \cdots + A_{k_l}^{(l)}\frac{t^{k_l-1}}{(k_l-1)!}\right]e^{+\sigma_l t}e^{+j\omega_l t}u(t) \\ &\left[\cancelto{A_1^{(l)*}}{A_1^{(k)}} + \cancelto{A_2^{(l)*}}{A_2^{(k)}}t + \cdots + \cancelto{A_{k_l}^{(l)*}}{A_{k_l}^{(k)}}\frac{t^{k_l-1}}{(k_l-1)!}\right]e^{+\sigma_l t}e^{-j\omega_l t}u(t) \\ &= 2\left[\left|A_1^{(l)}\right|\cos\left(\omega_l t + \angle A_1^{(l)}\right) + \left|A_2^{(l)}\right|t\cos\left(\omega_l t + \angle A_2^{(l)}\right) + \ldots\right. \\ &\left. + \left|A_{k_l}^{(l)}\right|\frac{t^{k_l-1}}{(k_l-1)!}\cos\left(\omega_l t + \angle A_{k_l}^{(l)}\right)\right]e^{+\sigma_l t}u(t) \end{aligned} \tag{9.29}$$

R The ROC of each term of the form $t^k e^{\sigma_l t}u(t)$ or $t^k\cos(\omega_l t + \phi)e^{\sigma_l t}u(t)$ is $\mathrm{Re}\{s\} > \sigma_l$. So the overall ROC is the intersection of all these ROCs, i.e. the RHP extending to the right of the rightmost pole.



9.8.3 How to Find the Coefficients of the PFE

Two methods:

- **Method 1:** Works well for simple PFEs (e.g. 2-3 terms), set up system of equations/unknowns and solve it.

■ Example 9.20

$$
\begin{aligned}
X(s) &= \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\
&= \frac{A(s+2)+B(s+1)}{(s+1)(s+2)} = \frac{(A+B)s+(2A+B)}{(s+1)(s+2)} \\
&\implies (A+B)s+(2A+B) = 1,\ \forall s \\
&\implies \left.\begin{aligned} A+B&=0 \\ 2A+B&=1 \end{aligned}\right\} \left.\begin{aligned} B&=-A \\ 2A-A&=1 \end{aligned}\right\} \implies \left.\begin{aligned} B&=-1 \\ A&=1 \end{aligned}\right\}
\end{aligned}
$$

■

- **Method 2:** Residues. We describe this method through examples for the two cases of non-repeated roots and repeated roots.

Case #1: The roots $s_1, s_2, \dots, s_p$ of the denominator polynomial $D(s)$ have all multiplicity 1. Then the partial fraction representation of $X(s)$ is:

$$X(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_p}{s-s_p} \tag{9.30}$$

Now, observe that

$$
\begin{aligned}
(s-s_i)X(s) &= \sum_{k=1}^{p} A_k \frac{(s-s_i)}{s-s_k} \\
&= A_i + \sum_{k=1,k\neq i}^{p} A_k \frac{(s-s_i)}{s-s_k} \\
\therefore \lim_{s\to s_i}(s-s_i)X(s) &= A_i + \lim_{s\to s_i} \sum_{k=1,k\neq i}^{p} \frac{s-s_i}{s-s_k} \\
&= A_i
\end{aligned}
$$

Thus, in general, when the roots of $D(s)$ are **not repeated**, we get

$$\boxed{A_i = \lim_{s \to s_i} X(s)\,(s - s_i)} \tag{9.31}$$

■ **Example 9.21** Express

$$X(s) = \frac{s^2+5}{s^3+7s^2+14s+8} = \frac{s^2+5}{(s+4)(s+1)(s+2)}$$

in partial fraction form. The zeros of the denominator are -4, -1, and -2 and each is of multiplicity 1. Thus we can write

$$X(s) = \frac{A_1}{s+4} + \frac{A_2}{s+1} + \frac{A_3}{s+2}$$

where

$$\begin{aligned} A_1 = \lim_{s \to -4} (s+4)X(s) &= \lim_{s \to -4} \frac{s^2+5}{(s+1)(s+2)} \\ &= \frac{7}{2} \end{aligned}$$

similarly

$$\begin{aligned} A_2 = \lim_{s \to -1} (s+1)X(s) &= \lim_{s \to -1} \frac{\left(s^2+5\right)}{(s+2)(s+4)} \\ &= 2 \\ A_3 = \lim_{s \to -2} (s+2)X(s) &= \lim_{s \to -2} \frac{s^2+5}{(s+1)(s+4)} \\ &= -\frac{9}{2} \end{aligned}$$

Therefore

$$\begin{aligned} X(s) &= \frac{7/2}{s+4} + \frac{2}{s+1} - \frac{9/2}{s+2} \\ \therefore\ x(t) &= \frac{7}{2}e^{-4t}u(t) + 2e^{-t}u(t) - \frac{9}{2}e^{-2t}u(t) \end{aligned}$$

■

The methodology is exactly the same when we have complex roots.

■ **Example 9.22** Complex Conjugate Poles:

$$\begin{aligned}
X(s) &= \frac{4s+10}{s^2+2s+2} \\
&= \frac{A_1^{(1)}}{s+(1+j)} + \frac{A_1^{(2)}}{s+(1-j)} \\
A_1^{(1)} &= \lim_{s\to-(1+j)} [s+(1+j)] \frac{4s+10}{s^2+2s+2} \\
&= \lim_{s\to-(1+j)} \frac{4s+10}{s+(1-j)} = 2+3j \\
A_1^{(2)} &= \left[A_1^{(1)}\right]^* = 2-3j \\
\therefore\ x(t) &= 2\,\mathrm{Re}\left\{A_1^{(1)} e^{-(1+j)t}\right\} u(t) \\
&= 2\left|A_1^{(1)}\right| e^{-t} \cos\left(-t+\angle A_1^{(1)}\right) u(t) \\
&= 2\sqrt{13} e^{-t} \cos\left(t-\tan^{-1}\frac{3}{2}\right) u(t)
\end{aligned}$$

■

Observe that for the **case of non-repeated roots** (i.e. roots of multiplicity 1) the coefficients A_i's in the partial fraction expansion

$$X(s) = \frac{N(s)}{D(s)} = \sum_{k=1}^{p} \frac{A_k}{s-s_k},\ D(s_k)=0,\ k=1,2,\ldots,p$$

can also be computed as follows:

$$\boxed{A_i = \frac{N(s_i)}{D'(s_i)}} \tag{9.32}$$

where the "prime" denotes differentiation with respect to s (i.e. d/ds).

Proof.

$$\begin{aligned}
A_i &= \lim_{s\to s_i} (s-s_i) X(s) \\
&= \lim_{s\to s_i} \frac{N(s)}{\frac{D(s)}{s-s_i}} = N(s_i) \lim_{s\to s_i} \frac{1}{\left[\frac{D(s)-D(s_i)}{s-s_i}\right]} \quad (\text{since } D(s_i)=0) \\
&= \frac{N(s_i)}{D'(s_i)}
\end{aligned}$$

□

■ **Example 9.23**

$$X(s) = \frac{s^2+5}{s^3+7s^2+14s+8}$$

The roots of the denominator are -4, -1, and -2. $N(s) = s^2 + 5$ and $D'(s) = 3s^2 + 14s + 14$.

$$\begin{aligned}
\therefore\ X(s) &= \frac{A_1}{s+4} + \frac{A_2}{s+1} + \frac{A_3}{s+2} \\
A_1 &= \frac{N(-4)}{D'(-4)} = \frac{21}{6} = \frac{7}{2} \\
A_2 &= \frac{N(-1)}{D'(-1)} = \frac{6}{3} = 2 \\
A_3 &= \frac{N(-2)}{D'(-2)} = \frac{9}{-2}
\end{aligned}$$

Note that above result is identical to that obtained earlier. ■

Case #2: One of the roots s_1, s_2, …, s_p of $D(s)$ is repeated (i.e., it has multiplicity > 1). Suppose, for example, the root s_1 has multiplicity k_1 and the remaining roots $\{s_2, s_3, \ldots, s_p\}$ are all of multiplicity 1. Then the form of the partial fraction expansion becomes

$$\begin{aligned}
X(s) = &\left[\frac{A_1^{(1)}}{s-s_1} + \frac{A_2^{(1)}}{(s-s_1)^2} + \cdots + \frac{A_{k_1}^{(1)}}{(s-s_1)^{k_1}}\right] \\
&+ \frac{A_1^{(2)}}{s-s_2} + \frac{A_1^{(3)}}{s-s_3} + \cdots + \frac{A_1^{(p)}}{s-s_p}
\end{aligned} \tag{9.33}$$

Now consider the terms in Eq. (9.33) containing the root s_1 of multiplicity k_1. It is easy to see that

$$\begin{aligned}
A_{k_1}^{(1)} &= \lim_{s\to s_1} (s-s_1)^{k_1} X(s) \\
A_{k_1-1}^{(1)} &= \lim_{s\to s_1} \frac{d}{ds}\left[(s-s_1)^{k_1} X(s)\right] \\
&\cdots \\
A_{k_1-r}^{(1)} &= \lim_{s\to s_1} \frac{1}{r!}\frac{d^r}{ds^r}\left[(s-s_1)^{k_1} X(s)\right] \\
&\cdots \\
A_1^{(1)} &= \lim_{s\to s_1} \frac{1}{(k_1-1)!}\frac{d^{k_1-1}}{ds^{k_1-1}}\left[(s-s_1)^{k_1} X(s)\right]
\end{aligned}$$

As before, for the remaining poles s_j of multiplicity 1, we have

$$A_1^{(j)} = \lim_{s\to s_j} (s-s_j) X(s)$$

■ Example 9.24

$$X(s) = \frac{s+2}{(s+1)^3(s+3)}$$

The roots are -1 and -3, and the multiplicites are 3 and 1, respectively.

$$\begin{aligned}
&= \frac{A_1^{(1)}}{s+1} + \frac{A_2^{(1)}}{(s+1)^2} + \frac{A_3^{(1)}}{(s+1)^3} \\
&+ \frac{A_1^{(2)}}{s+3} \\
A_3^{(1)} &= \lim_{s\to-1}(s+1)^3X(s) = \lim_{s\to-1}\frac{s+2}{s+3} = \frac{1}{2} \\
A_2^{(1)} &= \lim_{s\to-1}\frac{d}{ds}\left[(s+1)^3X(s)\right] = \lim_{s\to-1}\frac{d}{ds}\left[\frac{s+2}{s+3}\right] \\
&= \lim_{s\to-1}\frac{1}{(s+3)^2} = \frac{1}{4} \\
A_1^{(1)} &= \lim_{s\to-1}\frac{1}{2}\frac{d^2}{ds^2}\left[(s+1)^3X(s)\right] = \lim_{s\to-1}\frac{1}{2}\frac{-2}{(s+3)^3} = -\frac{1}{8} \\
A_1^{(2)} &= \lim_{s\to-3}(s+3)X(s) = \lim_{s\to-3}\frac{s+2}{(s+1)^3} = \frac{1}{8} \\
\therefore\ X(s) &= \frac{-1/8}{s+1} + \frac{1/4}{(s+1)^2} + \frac{1/2}{(s+1)^3} + \frac{1/8}{s+3} \\
\therefore\ x(t) &= -\frac{1}{8}e^{-t}u(t) + \frac{1}{4}te^{-t}u(t) + \frac{1}{4}t^2e^{-t}u(t) + \frac{1}{8}e^{-3t}u(t)
\end{aligned}$$

■

9.8.4 MATLAB and Partial Fraction Expansions

Doing partial fraction expansions by hand is **very tedious** and subject to careless errors. Why not let MATLAB do it for you?

Suppose you have some rational function $\frac{N(s)}{D(s)}$ where the order of the numerator polynomial $N(s)$ is less than or equal to the order of the denominator polynomial $D(s)$. Then

1. Write $N(s)$ and $D(s)$ as vectors, where the vectors consist of the polynomial coefficients in **descending order**.

   ■ Example 9.25

   $$\begin{aligned}
   \frac{N(s)}{D(s)} &= \frac{s+2}{s^4+6s^3+12s^2+10s+3} \\
   &= \frac{s+2}{(s+1)^3(s+3)}
   \end{aligned}$$

   ```
   N = [1,2];
   D = [1,6,12,10,3];
   ```

   ■

2. Use the MATLAB command `residue` to find the partial fraction expansion

   ```
   [R,p,K] = residue(N,D);
   ```

   This command returns values for the coefficients, R (a **vector** of values) in the partial fraction expansion; the roots, p (a vector of values) of $D(s)$; and K represents the quotient of the division between $N(s)$ and $D(s)$ (if the order of $N(s)$ is greater or equal to the order of $D(s)$); otherwise, $K = [\,]$.
   MATLAB Program:

```
» N = [1,2];
» D = [1,6,12,10,3];
» [c,p,k] = residue(N,D);
» c
[2]c =

        0.1250

       -0.1250

        0.2500

        0.500


» p
[2]p =

       -3.0000

       -1.0000

       -1.0000

       -1.0000


» k
[2]k =

       []
```

The roots of $D(s)$ are stored in p (with their multiplicities), and the coefficients corresponding to the roots of $D(s)$ are stored in c.

$$\therefore \frac{s+2}{(s+1)^3(s+3)} = \frac{0.125}{s+3} - \frac{0.125}{s+1} + \frac{0.25}{(s+1)^2} + \frac{0.5}{(s+1)^3}$$

Note that this result agrees with what we got earlier.

9.8.5 Other Useful MATLAB Commands

1. Consider two polynomials represented by two vectors U and V consisting of their coefficients in descending order. Then the command `conv(U,V)` multiplies the two polynomials and returns a vector of the coefficients in descending order.

   ■ Example 9.26

   ```
   [1]U = [1,2];
      V = [1,2,3];
      conv(U,V)
   ```

   $\}$ MATLAB program

   Result ⟶ `[1,4,7,6]`

   Thus $(s+2)(s^2+2s+3) = s^3+4s^2+7s+6.$ ■

2. Given a polynomial represented by a vector V consisting of its coefficients in descending order, the MATLAB command `roots` computes its roots.

   ■ Example 9.27

   ```
   [1]V = [1,-3,2];
      roots(V)
   ```

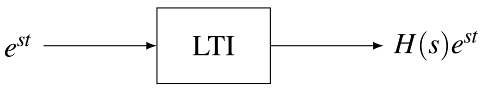
   $\}$ MATLAB program

   Result ⟶
   ```
   1
   2
   ```

Thus the roots of $s^2 - 3s + 2 = 0$ are 1 and 2. ■

9.9 System Transfer Function

Recall



for all $s \in ROC$ of $h(t)$.

Definition 9.8 — System Transfer Function.

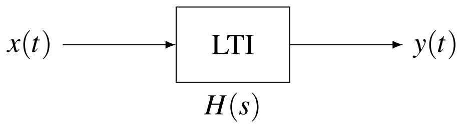
$$H(s) \triangleq \int_{0^-}^{+\infty} h(t)e^{-st}dt \tag{9.34}$$

is called the ***system transfer function*** (TF). If $j\omega \in ROC$ then

$$H(j\omega) = H(s)|_{s=j\omega} \tag{9.35}$$

is the ***frequency response function*** (FRF).

Using the convolution property of Laplace Transform, for any input $x(t)$ (with $x(t) = 0, t < 0$) we have



$$\begin{aligned} y(t) &= x(t) * h(t) \\ Y(s) &= X(s)H(s) \end{aligned} \tag{9.36}$$

■ **Example 9.28**

$$\begin{aligned} h(t) &= e^{-t}u(t) \\ x(t) &= e^{2t}u(t) \\ y(t) &= ? \end{aligned}$$

$$\begin{aligned} H(s) &= \frac{1}{s+1}, \ \mathrm{Re}\{s\} > -1 \\ X(s) &= \frac{1}{s-2}, \ \mathrm{Re}\{s\} > 2 \\ Y(s) &= H(s)X(s) = \frac{1}{(s+1)(s-2)} = \cdots = \frac{-1/3}{s+2} + \frac{1/3}{s-2}, \ \mathrm{Re}\{s\} > 2 \\ \implies y(t) &= -\frac{1}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t) \end{aligned}$$

■

We cannot solve this problem with Fourier Transform, since $x(t) = e^{2t}u(t)$ does not have a Fourier Transform (it is not absolutely integrable).

For systems described by LCCDE, e.g.

$$3y''(t)+2y'(t)+5y(t)=4x(t)+6x'(t)$$

the Transfer Function can be found easily as

$$H(s)=\frac{6s+4}{3s^2+2s+5}$$

as we did with Frequency Response Function.

■ **Example 9.29** Find the response of the system described by the LCCDE

$$y''(t)+3y'(t)+2y(t)=x(t)+x'(t)$$

when the input is $x(t)=u(t)$ **and** initial conditions are $y(0^-)=1$, $y'(0^-)=4$.

Apply Laplace Transform derivative property:

$$\begin{aligned}
&\left[s^2Y(s)-sy\left(0^-\right)-y'\left(0^-\right)\right]+3\left[sY(s)-y\left(0^-\right)\right]+2Y(s)\\
&=X(s)+\left[sX(s)-\cancelto{0}{x\left(0^-\right)}\right]\\
&\Longrightarrow\\
&\left(s^2+3s+2\right)Y(s)=(s+1)X(s)+(s+3)y\left(0^-\right)+y'\left(0^-\right)\\
&\Longrightarrow\ Y(s)=\underbrace{\underbrace{\frac{s+1}{s^2+3s+2}}_{TF}X(s)}_{ZSR}+\underbrace{\frac{(s+3)\cancelto{1}{y\left(0^-\right)}+\cancelto{-4}{y'\left(0^-\right)}}{s^2+3s+2}}_{ZIR}
\end{aligned}$$

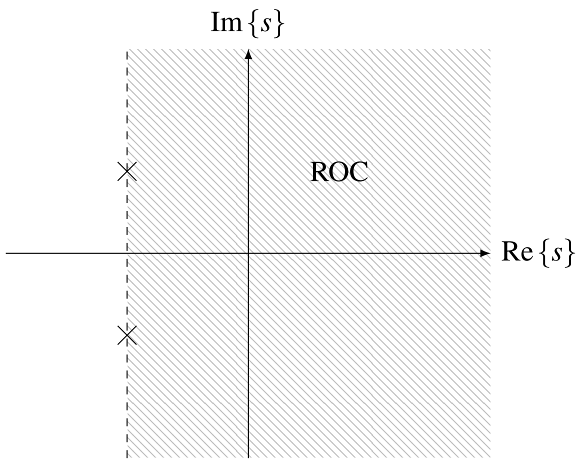
The ZSR depends only on $X(s)$ and the ZIR depends only on the initial conditions. ■

9.10 BIBO Stability

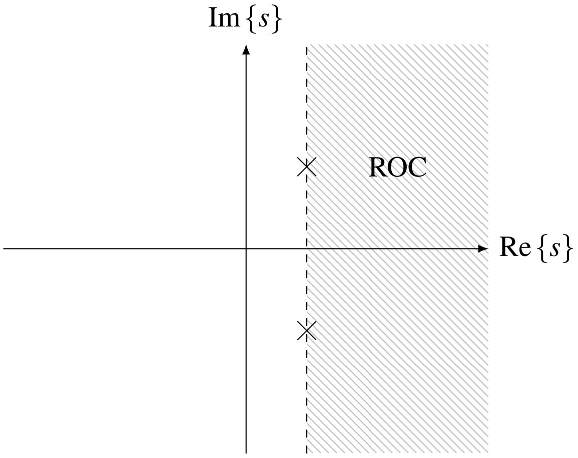
Consider a **causal** LTI system with impulse response $h(t)$ and transfer function $H(s)$. Recall that such a system is BIBO stable **if and only if** $h(t)$ is absolutely integrable, i.e.

$$\int_{0^-}^{+\infty}|h(t)|dt<\infty \iff H(j\omega)\text{ exists } \iff j\omega\in ROC\text{ of }H(s)$$

However we know that the ROC of $H(s)$ is a RHP extending to the right of the right-most pole of $H(s)$. So



or



So the $j\omega$ axis belongs to the ROC **if and only if** all poles of $H(s)$ are strictly to the left of the $j\omega$ axis. Thus,

A causal LTI system is BIBO stable **if and only if** all poles of $H(s)$ lie in the left half plane (LHP) (*strictly*), i.e.

$$\operatorname{Re}\{s_l\} < 0 \tag{9.37}$$

for all poles s_l of $H(s)$.

9.10.1 Simple Test for Stability

We can check if the poles (roots of $D(s)$) are on the LHP without finding the roots.

Motivation: The roots of a quadratic ($ax^2 + bx + c$) are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. So if a, b, $c > 0$ then both roots are negative!

Routh-Hurwitz Test:
For a 2nd and 3rd order polynomial

$\alpha_0 + \alpha_1 s + s^2$: if $\alpha_0,\ \alpha_1 > 0$
then all roots are on the LHP.
$\alpha_0 + \alpha_1 s + \alpha_2 s^2 + s^3$: if $\alpha_0,\ \alpha_1,\ \alpha_2 > 0$
AND $\alpha_1 \alpha_2 > \alpha_0$
then all roots are on the LHP.

9.11 Frequency Response Function *vs.* Transfer Function

Consider a causal LTI system with transfer function

$$H(s) = \frac{s + 2.5}{(s+2)(s+1+2j)(s+1-2j)}$$

The poles are -2, $-1 \pm 2j \implies$ stable $\implies$ FRF exists. The frequency response function is therefore

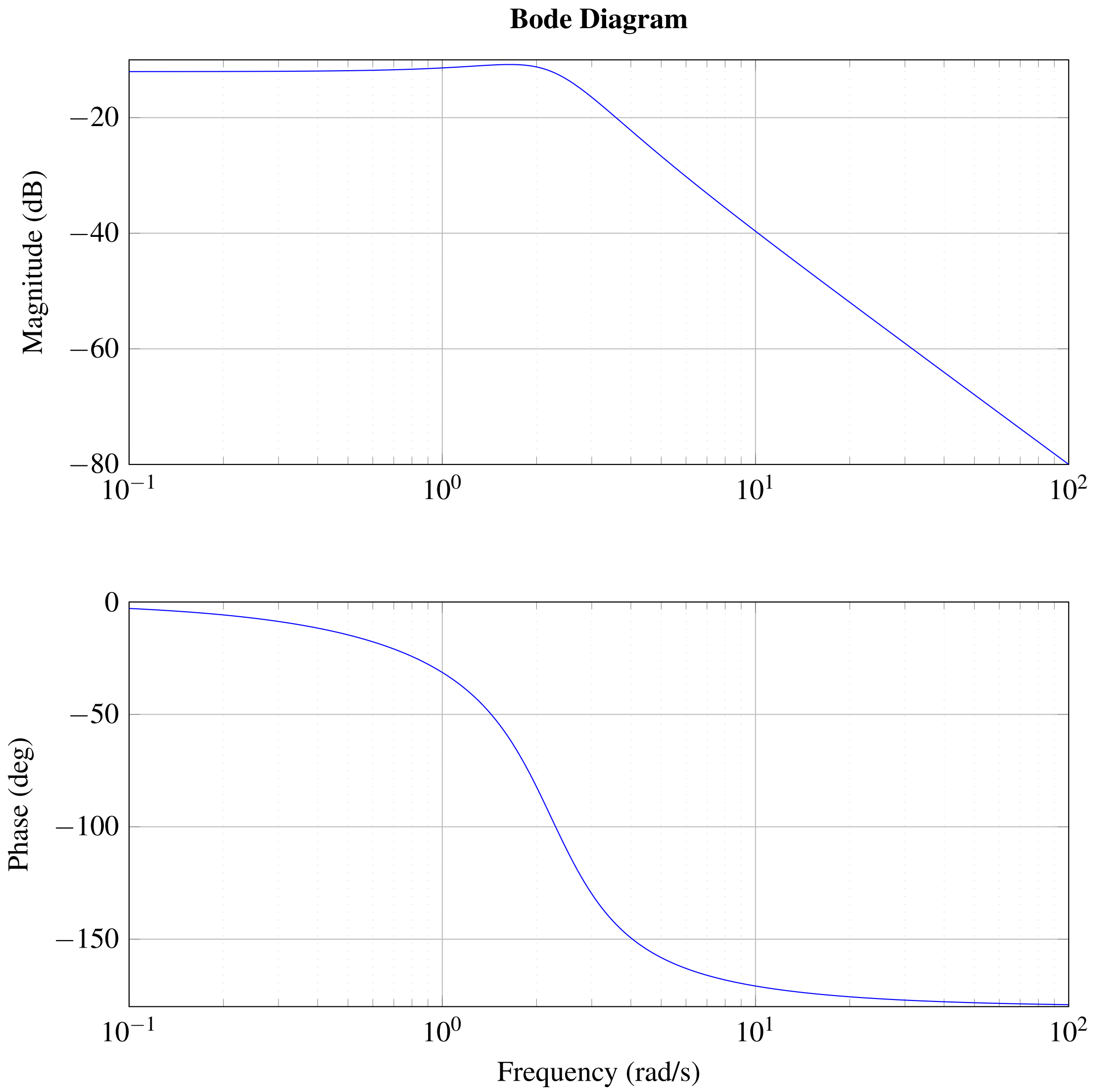
$$H(j\omega) = H(s)|_{s=j\omega} = \frac{j\omega + 2.5}{(j\omega+2)(j\omega+1+2j)(j\omega+1-2j)}$$

$H(s)$ and $H(j\omega)$ contain exactly the same information: from either we can extract $h(t)$ and this is all the information needed to fully characterize the LTI system.

Q: How do we visualize $H(j\omega)$?
A: Bode plot, i.e. plot $|H(j\omega)|$ and $\angle H(j\omega)$ with respect to ω.

■ **Example 9.30**

$$H(s) = \frac{s+2.5}{s^3+4s^2+9s+10}$$
$$H(j\omega) = \frac{2.5+j\omega}{(10-4\omega^2)+j(9\omega-\omega^3)}$$



Bode Diagram

■

- From this information we can answer questions of the sort: "What is the output $y(t)$ when the input $x(t) = A\cos(\omega_0 t + \phi)$?" (cosine-in cosine-out).

- However it is impossible to answer (by inspection) questions of the sort: "what is the output $y(t)$ when the input $x(t) = u(t)$.
  For this wee need to:

  - Evaluate $X(j\omega) = \mathscr{F}\{x(t)\}$
  - Evaluate $Y(j\omega) = H(j\omega)X(j\omega)$
  - Evaluate $y(t) = \mathscr{F}^{-1}\{Y(j\omega)\}$

  However, $H(s)$ can help us readily visualize the answer:

  $$Y(s) = H(s)X(s) = \frac{s+2.5}{(s+2)(s+1+2j)(s+1-2j)} \cdot \frac{1}{s}$$

  Poles: 0, -2, $-1 \pm 2j$

  $$\Longrightarrow y(t) = Au(t) + Be^{-2t}u(t) + 2|C|\cos(2t + \angle C)\,e^{-t}u(t)$$

9.12 Filter Design and Analysis Based on Poles and Zeros

Consider a system with m zeros (not necessarily distinct) and n poles (not necessarily distinct). We can write the system transfer function $H(s)$ as

$$H(s) = H_0 \frac{\prod_{k=1}^{m}(s - z_k)}{\prod_{l=1}^{n}(s - p_l)} \tag{9.38}$$

where $z_1, z_2, \ldots, z_m$ and $s_1, s_2, \ldots, s_n$ are the zeros and poles, respectively. IT follows that

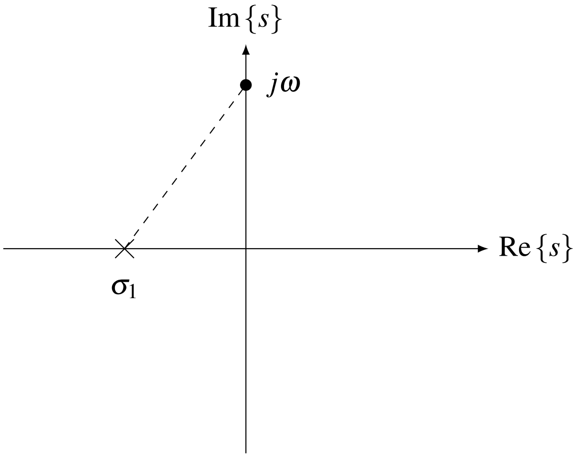
$$|H(j\omega)| = H_0 \frac{\prod_{k=1}^{m}|j\omega - z_k|}{\prod_{l=1}^{n}|j\omega - p_l|} \tag{9.39}$$

and

$$\angle H(j\omega) = \sum_{k=1}^{m} \angle(j\omega - z_k) - \sum_{l=1}^{n} \angle(j\omega - p_l) \tag{9.40}$$

Note that from a geometric perspective, a term of the form $|j\omega - s_0|$ is simply the distance between a point $j\omega$ on the $j\omega$-axis and some point $s = s_0$ in the $s-$plane. This observation can be used to design and analyze filters from their pole/zero plots.

Consider for example a filter $H(s)$ that has a single real-valued pole (and no zeros) located in the LHP as shown below.

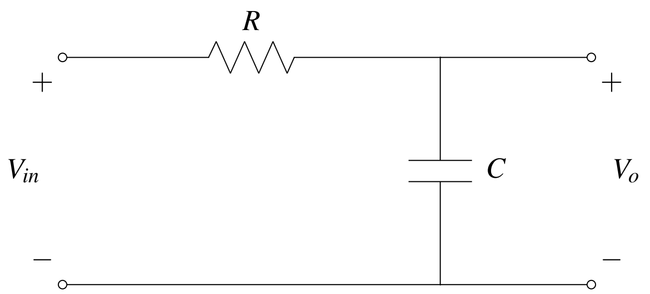


The length L_1 of the dashed line is $L_1 = |j\omega - \sigma_1|$. Then

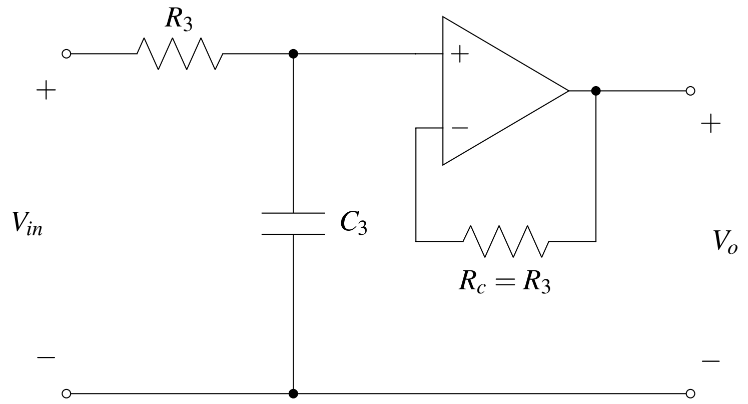
$$|H(j\omega)| \propto \frac{1}{L_1} = \frac{1}{|j\omega - \sigma_1|} = \frac{1}{\sqrt{\omega^2 + \sigma_1^2}} \tag{9.41}$$

Clearly (as seen from the above figure), L_1 is minimum for $\omega = 0$ and increases monotonically as $|\omega|$ increases from $\omega = 0$ to $\omega = \infty$. Consequently, this filter is a low-pass filter as illustrated below. It is also clear (geometrically from the figure above) that $|H(j\omega)|$ will decrease most rapidly with increasing $|\omega|$, when the pole $s = \sigma_1$ lies closest to the $j\omega$-axis. Thus the filter's bandwidth will be proportional to σ_1, i.e. the distance of this pole from the $j\omega$ axis.

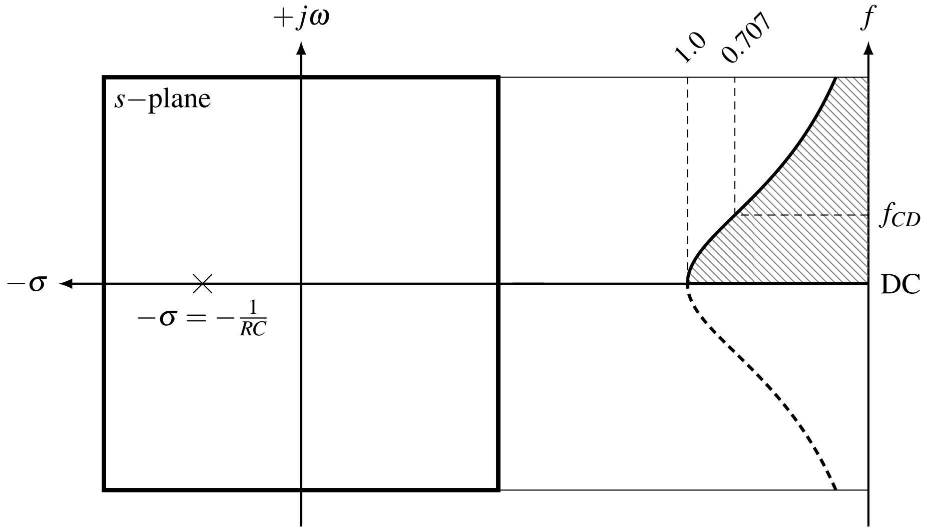
The same geometric arguments can be used to show that a system with a zero at the origin and a pole in the LHP on a real-axis is a high-pass filter (see figure below).



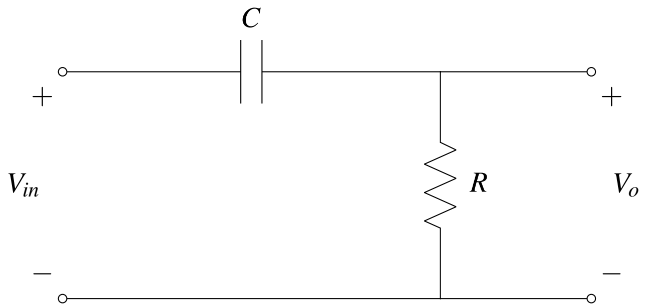
(a) Passive one pole low pass filter section



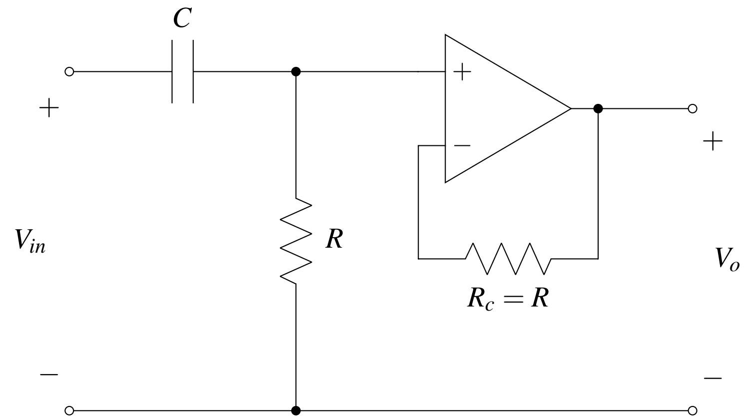
(b) Active one pole low pass filter section



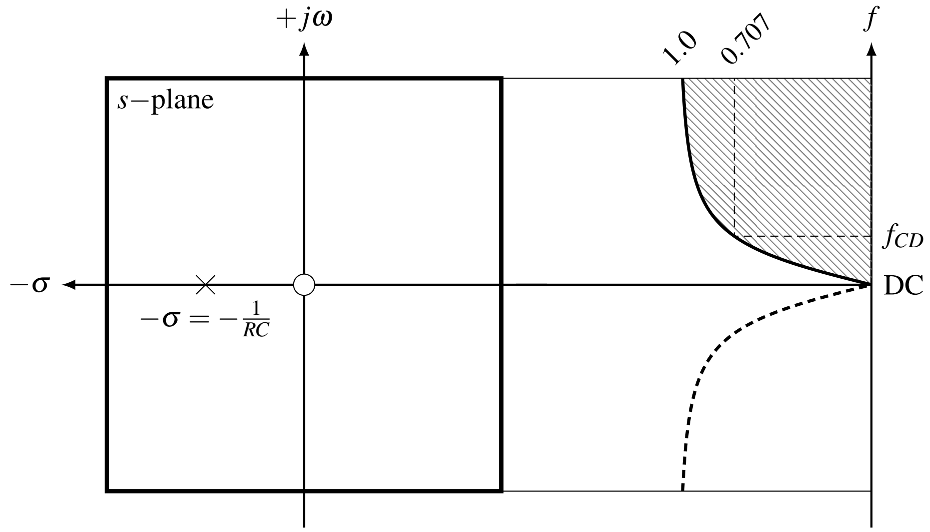
(c) Pole location for one pole low pass filter



(a) Passive one pole high pass filter section



(b) Active one pole high pass filter section



(c) Pole location for one pole high pass filter

From R. G. Irvine, **Operational Amplifiers** 3rd Edition, Prentice Hall, 1994.

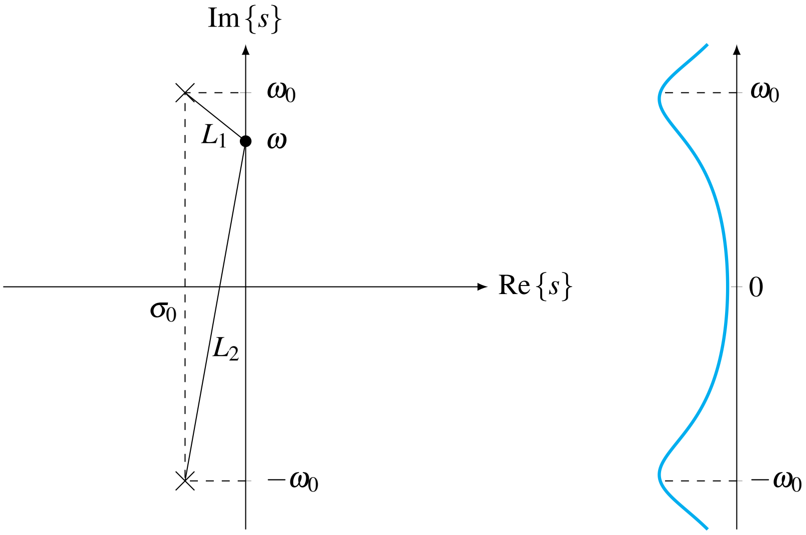
How about

$$H(j\omega) = \frac{1}{[j\omega - (\sigma_0 + j\omega_0)]\,[j\omega - (\sigma_0 - j\omega_0)]} \tag{9.42}$$

There is a complex conjugate pair of poles, $\sigma_0 \pm j\omega_0$.

$$|H(j\omega)| = \frac{1}{L_1 \cdot L_2} \tag{9.43}$$

This results in a BPF behavior.



$$|H(j\omega)| = \frac{1}{\sqrt{\left[(\omega - \omega_0)^2 + \sigma_0^2\right]\left[(\omega + \omega_0)^2 + \sigma_0^2\right]}} \tag{9.44}$$

Given a rational system transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

we can always factor it into ta product of first-order single pole or second-order complex-conjugate pole sections, i.e.

$$H(s) = H_0 \prod_{i=1}^{N} H_i(s) \tag{9.45}$$

where

$$H_i(s) = \frac{a_{i_1} s + a_{i_2}}{s + a_{i_3}} \quad 1^{\text{st}} \text{ order single-pole} \tag{9.46}$$

(a_{i_1} may be 0) or

$$H_i(s) = \frac{b_{i_1} s^2 + b_{i_2} s + b_{i_3}}{s^2 + b_{i_4} s + b_{i_5}} \quad 2^{\text{nd}}\text{-order pair of complex-conjugate poles} \tag{9.47}$$

(b_{i_1} and/or b_{i_2} may be 0). The 2nd order filter section directly above is known as a ***biquadratic filter***. Over the years a whole collection of circuits (see for example R. G. Irvine, **Operational Amplifiers** 3rd Edition, Prentice Hall, 1994) have been designed to implement 1st-order and biquadratic filter sections. These filter sections can then be cascaded to realize an arbitrary rational system transfer function

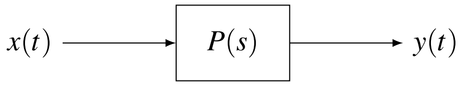
$$H(s) = \frac{N(s)}{D(s)}$$

10. Automatic Control

Introduction to control (viewgraphs).

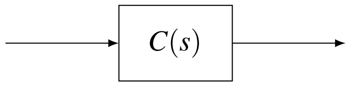
10.1 Components

- "Plant":



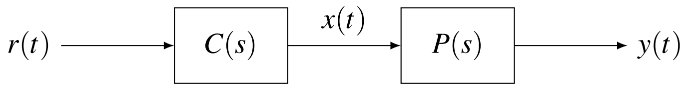
  i.e. the system that needs to be controlled.

- Controller:

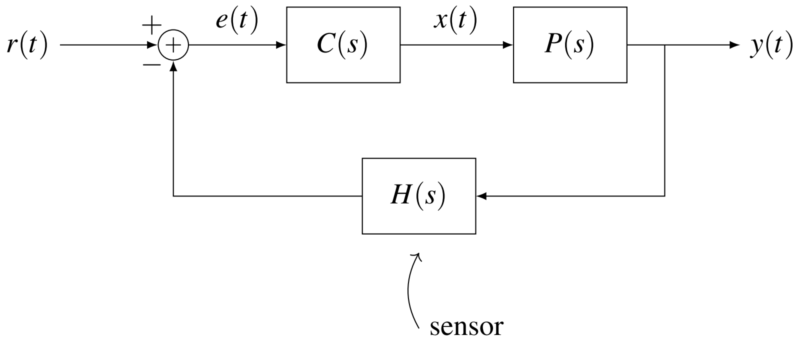


  A device to be designed.

- Open Loop:



- Closed Loop:



$$
\begin{aligned}
P_{OL}(s) &= C(s) \cdot P(s) \qquad (10.1) \\
P_{CL}(s) &= ?
\end{aligned}
$$

$$\left.\begin{aligned} &Y = P\cdot X = P\cdot C\cdot E \\ &E = R - HY \end{aligned}\right\} \implies Y = PC(R-HY)$$

$$\implies (1+PCH)Y = PCR \implies \frac{Y}{R} = \frac{PC}{1+PCH}$$

$$\implies \boxed{P_{CL}(s) = \frac{P(s)\cdot C(s)}{1+P(s)C(s)H(s)}} \tag{10.2}$$

Also,

$$\begin{aligned} E = R - HY &= R - HR\frac{PC}{1+PCH} \\ &= R\left(1 - \frac{HPC}{1+PCH}\right) = R\frac{1}{1+PCH} \end{aligned}$$

$$\implies \boxed{\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)H(s)}} \tag{10.3}$$

10.2 Design Process

Given $P(s)$ we study different types of open/close-loop control $C(s)$ with the following design goals (in order of significance):

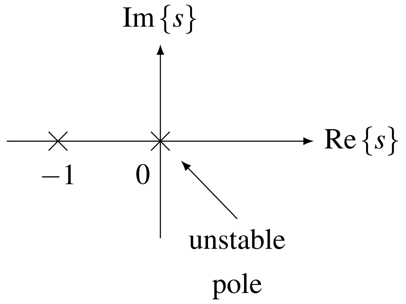
(a) Stabilize the plant.

(b) Reduce the steady-state error $e(t)$ between the command $r(t)$ and the system output $y(t)$ for different "test" signals, e.g. $u(t)$, $tu(t)$, $t^2u(t)$.

(c) Reach the steady-state as soon as possible.

■ **Example 10.1** ***Control of DC Motor***:
Model:

$$y'' + y' = x \implies P(S) = \frac{1}{s(s+1)}$$



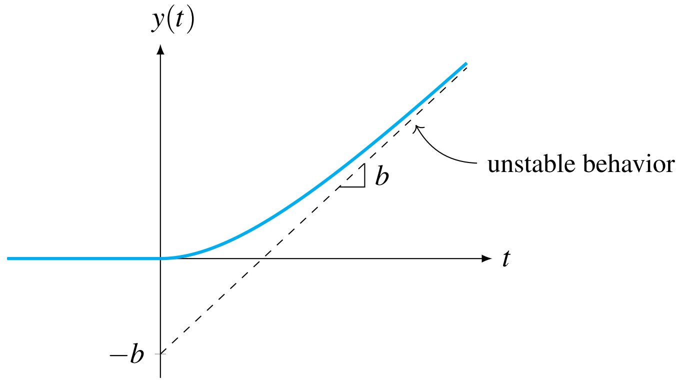
(10.4)

where $x(t)$ is the input voltage and $y(t)$ is the phase of the rotor (not angular velocity).

$$\begin{aligned} x(t) = bu(t) &\implies X(s) = \frac{b}{s} \\ Y(s) = X(s)P(s) &= \frac{b}{s}\frac{1}{s(s+1)} = -\frac{b}{s} + \frac{b}{s^2} + \frac{b}{s+1} \end{aligned}$$

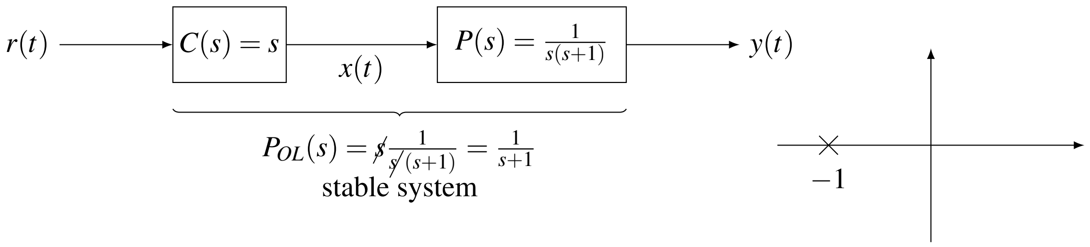
Therefore

$$\begin{aligned} y(t) &= -bu(t) + btu(t) + be^{-t}u(t) \\ &= b(t-1)u(t) + be^{-t}u(t) \end{aligned}$$



■

10.2.1 Open Loop Differential Control



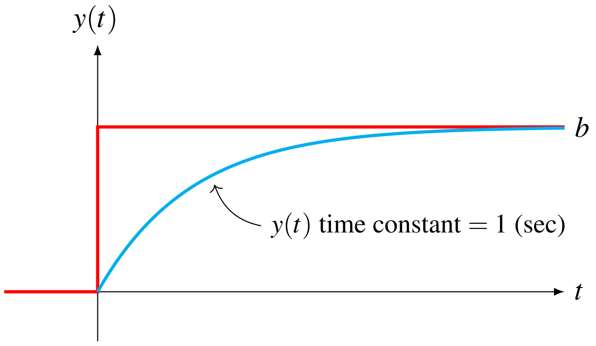
$$\underbrace{}_{P_{OL}(s) = \cancel{s}\frac{1}{\cancel{s}(s+1)} = \frac{1}{s+1}}$$

stable system

$$r(t) = bu(t) \implies R(s) = \frac{b}{s}$$

$$Y(s) = R(s)C(s)P(s) = \frac{b}{s}\frac{1}{s+1} = \frac{b}{s} - \frac{b}{s+1}$$

$$\implies y(t) = bu(t) - be^{-t}u(t) = b\left(1 - e^{-t}\right)u(t)$$

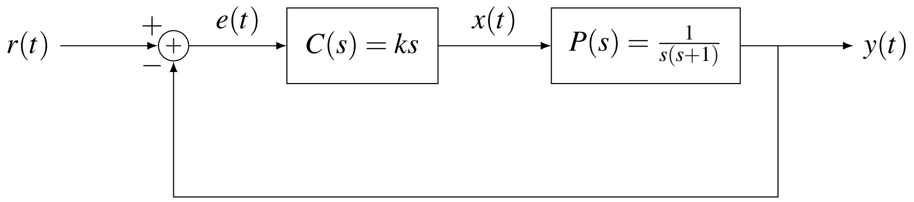


The steady state (ss) error is 0. This can also be found by the Final Value Theorem:
If $\lim_{t\to\infty} y(t)$ exists (check $Y(s)$)

Then

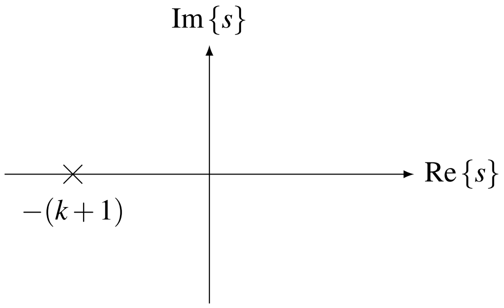
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}}\frac{1}{s+1} = b$$

10.2.2 Closed Loop Differential Control



$$C(s) = k \cdot s \quad \text{(we will design the value of } k\text{)}$$

$$P_{CL}(s) = \frac{ks\frac{1}{s(s+1)}}{1+ks\frac{1}{s(s+1)}} = \frac{ks}{s(s+1)+ks} = \frac{k}{s+k+1}$$



For stability,

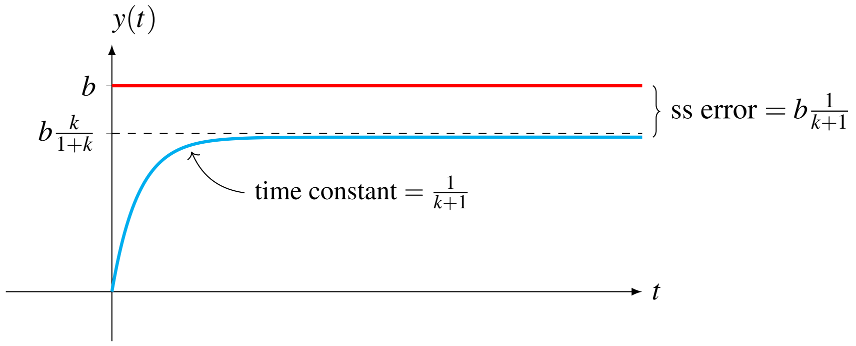
$$k+1>0 \Longleftrightarrow \boxed{k>-1}$$

Test step response:

$$r(t) = bu(t) \implies R(s) = \frac{b}{s}$$

$$Y(s) = R(s)P_{CL}(s) = \frac{b}{s}\frac{k}{s+k+1} = \frac{\frac{bk}{1+k}}{s} - \frac{\frac{bk}{1+k}}{s+k+1}$$

$$\implies y(t) = b\frac{k}{1+k}\left(1-e^{-(k+1)t}\right)u(t)$$



We can also find the steady-state error from

$$E(s) = R(s)\frac{1}{1+k\cancel{s}\frac{1}{\cancel{s}(s+1)}} = R(s)\frac{s+1}{s+k+1}$$
$$= \frac{b}{s}\frac{s+1}{s+k+1} = \frac{A}{s} + \frac{B}{s+k+1}$$

$\lim\limits_{t\to\infty} e(t)$ exists since $\frac{A}{s}$ results in $Au(t)$ and $\frac{B}{s+k+1}$ results in $Be^{-(k+1)t}u(t)$. So

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}}\frac{s+1}{s+k+1} = \frac{b}{k+1}$$

So if we select $k \gg 1$ then in addition to stability, we get

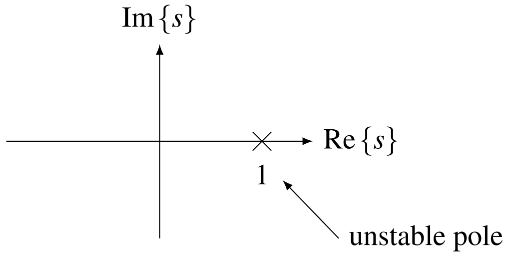
i) Negligible steady-state error $\frac{b}{k+1}$

ii) Small time constant $\frac{1}{k+1}$

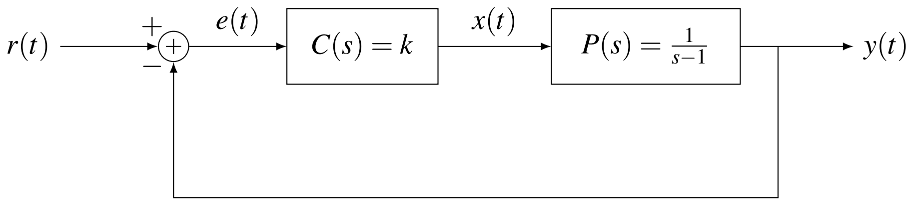
For these reasons, closed-loop control offers benefits over open-loop.

■ **Example 10.2** ***Proportional Control***:

$$P(s) = \frac{1}{s-1}$$



$C(s) = k$ (proportional control)



$$P_{cl}(s) = \frac{k\frac{1}{s-1}}{1+k\frac{1}{s-1}} = \frac{k}{s+k-1}$$

$P_{cl}(s)$ has a pole $-(k-1)$, so for stability we require $\boxed{k>1}$.

Test signal $r(t) = b \cdot u(t) \implies R(s) = \frac{b}{s}$:

$$Y(s) = R(s)P_{cl}(s) = \frac{b}{s}\frac{k}{s+k-1}$$

Apply FVT: $\lim\limits_{t\to\infty} y(t)$ exists (see $Y(s)$).

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}}\frac{k}{s+k-1} = b\frac{k}{k-1}$$

which can be made $\approx b$ for $k \gg 1$.

Test signal $r(t) = b \cdot t \cdot u(t)$ (ramp):

$$
\begin{aligned}
Y(s) &= \frac{b}{s^2}\frac{k}{s+k-1} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+k-1} \\
y(t) &= Au(t) + \frac{bk}{k-1}tu(t) + \frac{bk}{(k-1)^2}e^{-(k-1)t}u(t)
\end{aligned}
$$

where $B = \frac{bk}{k-1}$ and $C = \frac{bk}{(-k+1)^2} = \frac{bk}{(k-1)^2}$. We cannot apply the Final Value Theorem because of the second term. We can try to work directly with $E(s)$:

$$
E(s) = \frac{b}{s^2}\frac{1}{1+k\frac{1}{s-1}} = \frac{b}{s^2}\frac{s-1}{s+k-1}
$$

Even for $e(t)$ we cannot apply the Final Value Theorem because of the $\frac{1}{s^2}$ term in the PFE of $E(s)$. We can directly evaluate $e(t)$:

$$
\begin{aligned}
E(s) &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+k-1} \\
e(t) &= (A+Bt)u(t) + Ce^{-(k-1)t}u(t)
\end{aligned}
$$

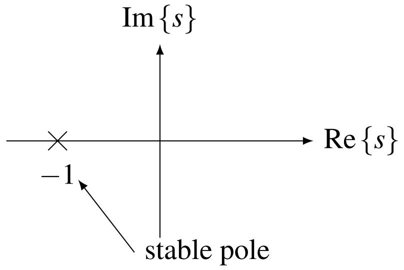
$$
\begin{array}{ccc}
\nearrow & \uparrow & \uparrow \\
\frac{bk}{(k-1)^2} & -\frac{b}{k-1} & -\frac{bk}{(k-1)^2}
\end{array}
$$

$$
\overset{\text{ss}}{\approx} (A+Bt)u(t)
$$

which increases indefinitely. ■

■ **Example 10.3** ***Integral Control***

$$
P(s) = \frac{1}{s+1}
$$



$$
\begin{aligned}
C(s) &= \frac{k}{s} \quad \text{(integral control)} \\
P_{cl}(s) &= \frac{\frac{k}{s}\frac{1}{s+1}}{1+\frac{k}{s}\frac{1}{s+1}} = \frac{k}{s(s+1)+k} = \frac{k}{\underbrace{s^2+s+k}_{2^{\text{nd}}\text{ order system}}}
\end{aligned}
$$

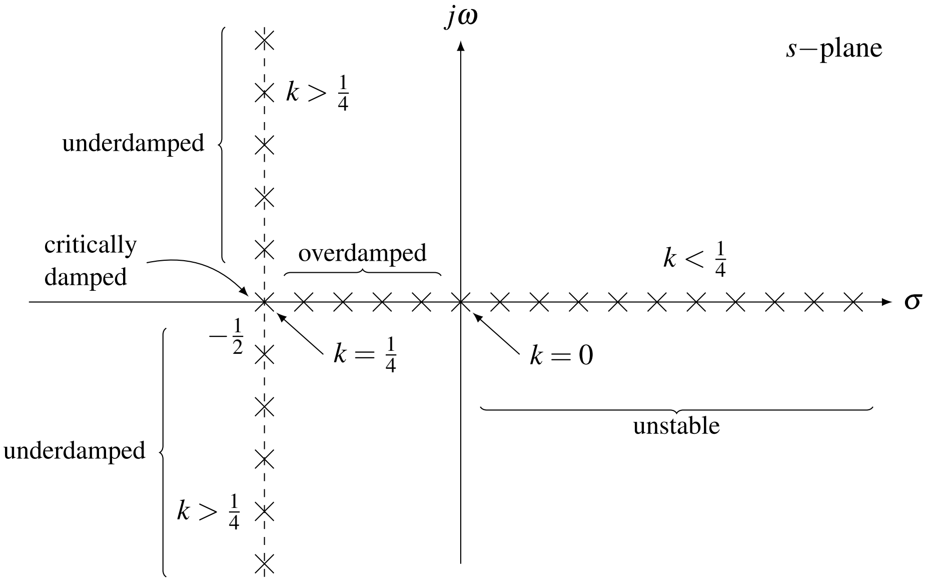
For stability (use Routh-Hurwitz test) we need $\boxed{k > 0}$.

BTW: What is the locus of the largest pole for different values of k?

$$
p = \frac{-1+\sqrt{1-4k}}{2}
$$

- For $k < \frac{1}{4}$ pole is real

- For $k = \frac{1}{4}$ double real pole $= -\frac{1}{2}$
- For $k > \frac{1}{4}$ two complex conjugate poles with real part $= -\frac{1}{2}$



Test signal $r(t) = b \cdot u(t) \implies R(s) = \frac{b}{s}$:

$$Y(s) = \frac{b}{s}\frac{k}{s^2+s+k}$$

We can apply the Final Value Theorem.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}}\frac{k}{s^2+s+k} = b$$

So: **zero steady-state error**

Let's try another test signal:

$$r(t) = b \cdot t \cdot u(t) \implies R(s) = \frac{b}{s^2}$$

$\lim\limits_{t\to\infty} y(t)$ does not exist so we cannot apply the Final Value Theorem. We better work directly with $E(s)$:

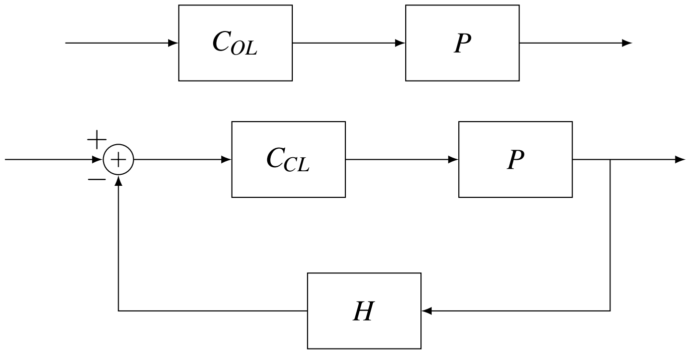
$$\begin{aligned} E(s) &= \frac{b}{s^2}\frac{1}{1+\frac{k}{s}\frac{1}{s+1}} = \frac{b}{\cancel{s^2}}\frac{\cancel{s}(s+1)}{s(s+1)+k} \\ &= \frac{b}{s}\frac{s+1}{s^2+s+k} \end{aligned}$$

$\lim\limits_{t\to\infty} e(t)$ exists so we can use the Final Value Theorem.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}}\frac{s+1}{s^2+s+k} = \frac{b}{k}$$

So if we select $\boxed{k \gg 1}$ the steady state error is ≈ 0. ■

10.3 Open Loop vs. Closed Loop Control



$$P_{OL} = C_{OL} \cdot P$$

$$P_{CL} = \frac{C_{CL}P}{1 + C_{CL}PH}$$

$$P_{OL} = P_{CL} \implies C_{OL} = \frac{C_{CL}}{1 + C_{CL}PH}$$

So for any $C_{CL}(s)$ we cand find a $C_{OL}(s)$ such that the overall open-loop system is the same as the overall closed-loop system, as long as:

- $P(s)$ is accurately known
- $P(s)$ is not time-varying
- $P(s)$ not subject to external disturbances

If any of these three conditions is not satisfied, then closed-loop offers advantages (more robust):

- Makes closed-loop system insensitive to variations of $P(s)$
- Makes system insensitive to disturbances

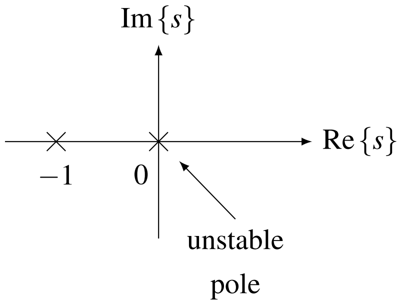
Disadvantages of closed-loop:

- Increased complexity
- Reduced gain (see example)

■ **Example 10.4 Open-loop *vs.* Closed-loop with Variations of $P(s)$**

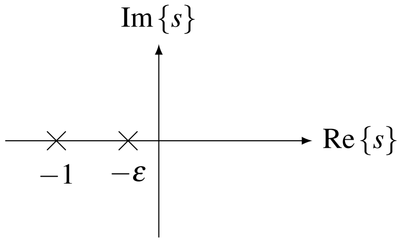
Consider the DC motor example:

$$P(S) = \frac{1}{s(s+1)}$$



We use differential control (as before). Assume that the **actual plant** has

$$\hat{P}(S) = \frac{1}{(s+\varepsilon)(s+1)}$$



for some small value ε (due to "aging" of the components).

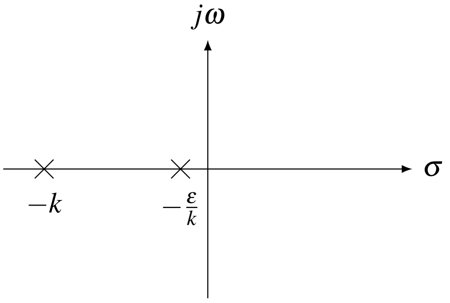
Closed-Loop:

$$\begin{aligned}
C(s) &= k\cdot s\\
P_{CL}(s) &= \frac{ks\frac{1}{(s+\varepsilon)(s+1)}}{1+ks\frac{1}{(s+\varepsilon)(s+1)}} = \frac{ks}{ks+(s+\varepsilon)(s+1)} = \frac{ks}{s^2+(k+1+\varepsilon)s+\varepsilon}\\
R(s) &= \frac{b}{s}\\
Y(s) &= \frac{b}{\cancel{s}}\frac{k\cdot\cancel{s}}{s^2+(k+1+\varepsilon)s+\varepsilon}
\end{aligned}$$

Stability:

$$\left|\begin{array}{l}\varepsilon > 0\\ k+1+\varepsilon > 0\end{array}\right.$$

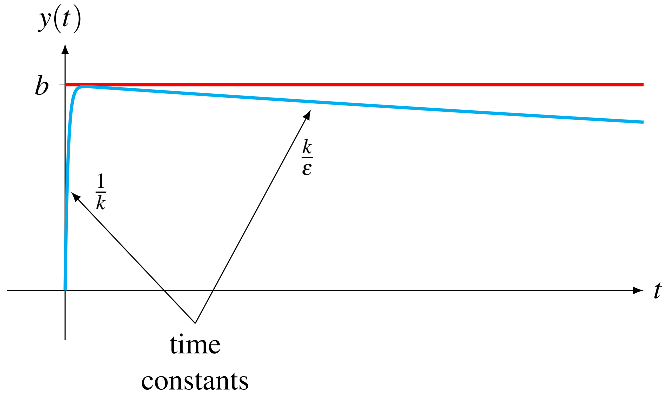
(assume $k \gg 1$ and $0 < \varepsilon \ll 1$)
Poles $\approx -\frac{\varepsilon}{k}, -k$



$$Y(s) \approx \frac{A}{s+\frac{\varepsilon}{k}} - \frac{A}{s+k}$$

where $A = b\frac{k^2}{k^2-\varepsilon}$.

$$\Longrightarrow\; y(t) \approx b\frac{k^2}{k^2-\varepsilon}\left(e^{-\frac{\varepsilon}{k}t} - e^{-kt}\right)u(t) \approx b\left(e^{-\frac{\varepsilon}{k}t} - e^{-kt}\right)u(t)$$

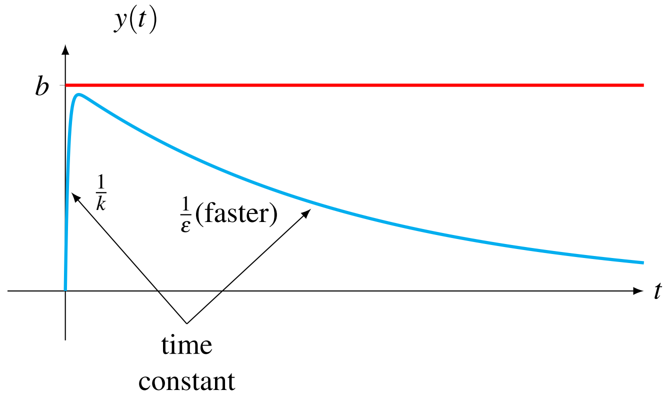


■

Open Loop:

Choose the open loop controller so that it is the "equivalent" open loop controller to the above closed loop controller (assuming a perfect system) (recall $C_{OL} = \frac{C_{CL}}{1+C_{CL}PH}$)

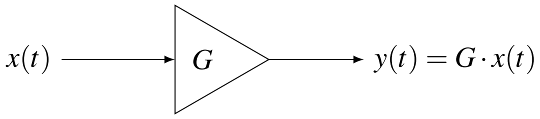
$$
\begin{aligned}
C_{OL}(s) &= \frac{ks}{1+ks\frac{1}{s(s+1)}} = \frac{ks(s+1)}{s+k+1} \\
P_{OL}(s) &= \frac{ks}{(s+k+1)(s+\varepsilon)} \approx \frac{ks}{(s+k)(s+\varepsilon)} \qquad (\text{for } k \gg 1) \\
R(s) &= \frac{b}{s} \\
Y(s) &= R(s)P_{OL}(s) \approx \frac{b}{\cancel{s}}\frac{k\cancel{s}}{(s+k)(s+\varepsilon)} = \frac{\frac{bk}{k-\varepsilon}}{s+\varepsilon} - \frac{\frac{bk}{k-\varepsilon}}{s+k} \\
y(t) &\approx \frac{bk}{k-\varepsilon}\left(e^{-\varepsilon t} - e^{-kt}\right)u(t) \approx b\left(e^{-\varepsilon t} - e^{-kt}\right)u(t)
\end{aligned}
$$



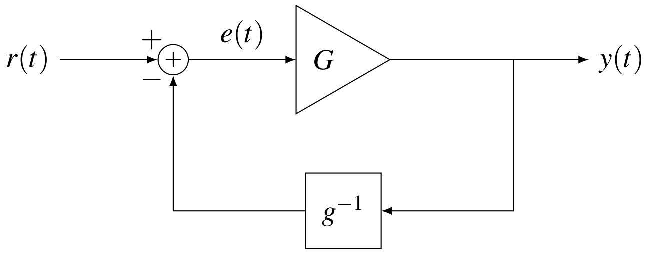
So, comparing open-loop and closed-loop controller implementations, the performance of closed-loop is better when an error ε is present in the description of the plant. Specifically we see that closed-loop decays to zero slower by a factor of k.

■ **Example 10.5** ***Robustness to internal parameters***

Consider a simpler amplifier system with large but variable gain G:

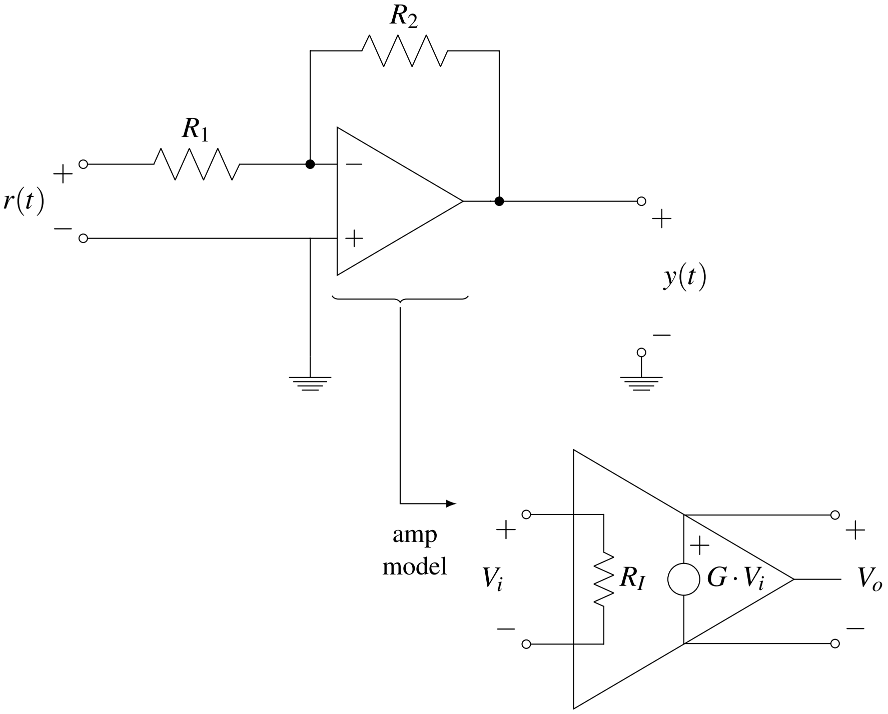


where $G \gg 1$ (but due to manufacturing issues the value of G is not well known). Use closed-loop control:



$$\left.\begin{aligned} y(t) &= e(t) \cdot G \\ e(t) &= r(t) - g^{-1} y(t) \end{aligned}\right\} \implies y(t) = r(t) \underbrace{\frac{G}{1+g^{-1}G}}_{\text{gain}};$$

$G \gg 1 \implies \text{gain} \approx g$. So the overall amplifier gain is not determined by (the unrealiable) G but by the (reliable) parameter g. This is exactly what happens when you use an op-amp:

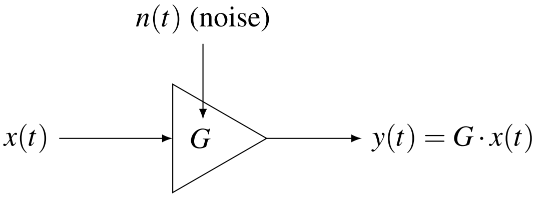


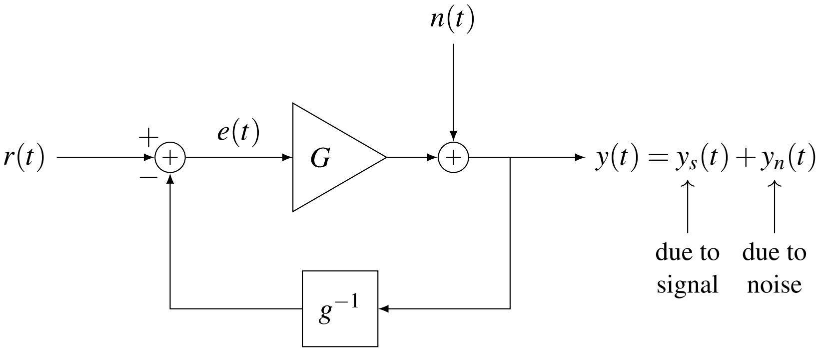
$$y(t) \overset{G \gg 1}{\approx} -\frac{R_2}{R_1} \quad \text{(does not depend on } G\text{)}$$

■

■ Example 10.6 ***Effect of Noise***

What is the effect of noise generated from the electronic components or as an outside disturbance?





We saw that $y_s(t) = r(t)\frac{G}{1+g^{-1}G} \underset{G\gg 1}{\approx} gr(t)$. For $y_n(t)$ we set $r(t) = 0$ and

$$\left.\begin{aligned} y_n(t) &= n(t) + G\cdot e(t)\\ e(t) &= -g^{-1}y_n(t)\end{aligned}\right\} \implies y_n(t) = \frac{n(t)}{1+g^{-1}G} \underset{G\gg 1}{\approx} \frac{g}{G}n(t)$$

So overall (for $G \gg 1$)

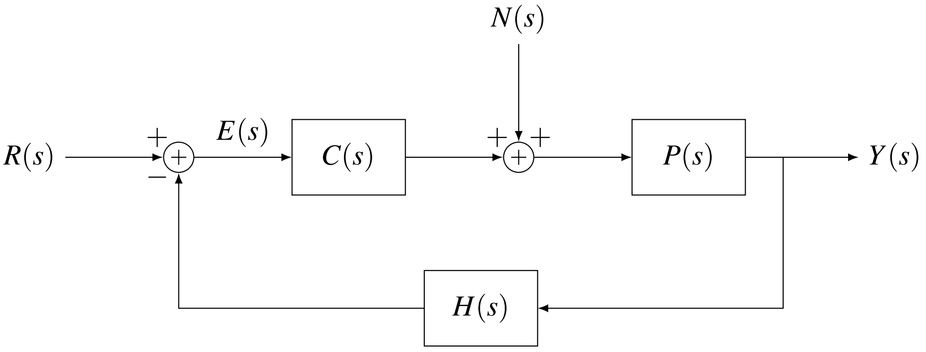
$$y(t) \approx \underset{\substack{\uparrow\\ \text{Gain}\\ \text{"stabilizes"}\\ \text{to a value}\\ \text{independent}\\ \text{of } G}}{g\cdot r(t)} + \underset{\substack{\uparrow\\ \text{noise is reduced}\\ \text{by a factor of } G}}{\frac{g}{G}n(t).} \tag{10.5}$$

However if we had used an open loop control which was equivalent to the closed-loop control we would have gotten $C_{OL}(s) = g/(g+G) \approx g/G$ and the overall response of the system would have been

$$y(t) \approx \underset{\substack{\uparrow\\ \text{Gain}\\ \text{"stabilizes"}\\ \text{to a value}\\ \text{independent}\\ \text{of } G}}{g\cdot r(t)} + \underset{\substack{\uparrow\\ \text{noise is not}\\ \text{reduced}}}{n(t)} \tag{10.6}$$

■

In general:



$$\left.\begin{aligned} Y_s(s) &= R(s)\frac{CP}{1+CPH}\\ Y_N(s) &= N(s)\frac{P}{1+CPH}\end{aligned}\right\} \implies$$

$$\begin{aligned} Y(s) &= R(s)\frac{CP}{1+CPH} + N(s)\frac{P}{1+CPH}\\ &\underset{C(s)=k\gg 1}{\overset{H(s)=1}{=}} R(s) + \frac{1}{k}N(s). \end{aligned} \tag{10.7}$$

However if we had used an open loop control which was equivalent to the closed-loop control we would have gotten $C_{OL} = \frac{C}{1+CPH} \approx 1/P$ and the overall response of the system would have been

$$\begin{aligned} Y(s) &= R(s)\frac{C}{1+CPH}P + N(s) \\ &\overset{H(s)=1}{\underset{C(s)=k\gg 1}{=}} R(s) + N(s). \end{aligned} \tag{10.8}$$

10.4 Summary

We use common controllers, such as Proportional-Integral-Differential (PID)

$$C(s) = K_P + \frac{K_I}{s} + K_D \cdot s \tag{10.9}$$

where K_P, K_I, and K_D are design parameters, with the following objectives:

- Stabilize the close-loop system
- Test system with inputs

  $$u(t),\ tu(t),\ t^2u(t),\ \ldots$$

  and make sure that

  - steady-state error is small
  - transients die out and fast enough